SAITO-KUROKAWA LIFTS AND APPLICATIONS TO THE BLOCH-KATO CONJECTURE

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Abstract. Let f be a newform of weight $2k-2$ and level 1. In this paper we provide evidence for the Bloch-Kato conjecture for modular forms. We demonstrate an implication that under suitable hypotheses if $\varpi \mid L_{\mathrm{alg}}(k,f)$ then $p \mid \# \mathrm{H}_f(\mathbb{Q}, W_f(1-k))$ where p is a suitably chosen prime and ϖ a uniformizer of a finite extension $K/\mathbb{Q}_p$. We demonstrate this by establishing a congruence between the Saito-Kurokawa lift F_f of f and a cuspidal Siegel eigenform G that is not a Saito-Kurokawa lift. We then examine what this congruence says in terms of Galois representations to produce a non-trivial p-torsion element in $\mathrm{H}^1_f(\mathbb{Q}, W_f(1-k))$.

1. Introduction

Let f be a newform of weight $2k-2$ and level 1. The Bloch-Kato conjecture for modular forms roughly states that the special values of the L-function associated to f should measure the size of the corresponding Selmer groups. In this paper we will demonstrate under suitable hypotheses that if $\varpi \mid L_{\mathrm{alg}}(k,f)$, then $p \mid \# \mathrm{H}^1_f(\mathbb{Q}, W_f(1-k))$ where p is a suitably chosen prime and ϖ is a uniformizer of a finite extension $K/\mathbb{Q}_p$. For a precise statement see Theorem 8.4.

The general outline of the method of proof of Theorem 8.4 goes back to Ribet's proof of the converse of Herbrand's theorem ([31]), which was then extended by Wiles in his proof of the main conjecture of Iwasawa theory for totally real fields ([52]). The method used by Ribet and Wiles is as follows. Given a positive integer k and a primitive Dirichlet character χ of conductor N so that $\chi(-1) = (-1)^k$, one has an associated Eisenstein series $E_{k,\chi}$. For a prime $p \nmid N$, one can show that there is a cuspidal eigenform g of weight k and level M with $N \mid M$ so that $g \equiv E_{k,\chi} (\mathrm{mod}\, \mathfrak{p})$ for some prime $\mathfrak{p} \mid p$. This congruence is used to study the residual Galois representation of g. It is shown that $\overline{\rho}_{g,\mathfrak{p}} \simeq \begin{pmatrix} 1 & * \\ 0 & \chi\omega^{k-1} \end{pmatrix}$ is non-split where ω is the reduction of the p-adic cyclotomic character. This allows one to show that $*$ gives a non-zero cohomology class in $\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}, \chi^{-1}\omega^{1-k})$.

For our purposes, the character in the Ribet/Wiles' method will be replaced with a newform f of weight $2k-2$ and level 1. Associated to f we have its Saito-Kurokawa lift F_f, our replacement for the Eisenstein series $E_{k,\chi}$. Our goal is to find a cuspidal Siegel eigenform G that is not a Saito-Kurokawa lift so that the eigenvalues of G are congruent modulo ϖ to those of F_f. We are able to produce such a G by exploiting the explicit nature of the Saito-Kurokawa correspondence. Also central to producing G is an inner product relation due to Shimura ([43]). In order to assure the G we construct is not a Saito-Kurokawa lift, we are forced to

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act on G with a particular Hecke operator that kills all Saito-Kurokawa lifts other then F_f. It is in this step that we must insert the hypothesis that f is ordinary at p. It appears this is merely a technical restriction that we hope to remove in a subsequent paper. For the precise statement of the congruence see Theorem 6.5.

Once we have produced a congruence modulo ϖ between the Hecke eigenvalues of F_f and G, we study the associated 4-dimensional Galois representations. Again we use the explicit nature of the Saito-Kurokawa correspondence to conclude that $\overline{\rho}_{F_f} \simeq \omega^{k-2} \oplus \overline{\rho}_f \oplus \omega^{k-1}$. Using our congruence we are able to determine that $\overline{\rho}_G^{\mathrm{ss}} \simeq \overline{\rho}_{F_f}$. From this we deduce the form of $\overline{\rho}_G$ by adapting arguments in [31] to the 4-dimensional case and applying results of [45] on the necessary shape of ρ_G. Some elementary arguments using class field theory allow us to conclude that we have a non-zero torsion element of the Selmer group $\mathrm{H}^1_f(\mathbb{Q}, W_f(1-k))$. We conclude with a non-trivial numerical example of Theorem 8.4 with $p = 516223$ and f of weight 54.

While this paper only deals with the case of full level, it is anticipated that similar results hold true for odd square-free level. We hope to treat the case of odd square-free level in a subsequent paper.

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2. Notation and definitions

In this section we fix notation and definitions that will be used throughout this paper.

Denote the adeles over $\mathbb{Q}$ by $\mathbb{A}$. We let $\mathbf{f}$ denote the finite set of places. For p a prime number, we fix once and for all compatible embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, and $\overline{\mathbb{Q}}_p \hookrightarrow \mathbb{C}$. Let ε_p be the p-adic cyclotomic character $\varepsilon_p : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_1(\mathbb{Z}_p)$. Recall that ε_p is unramified away from p and one has $\varepsilon_p(\mathrm{Frob}_\ell) = \ell$ for $\ell \neq p$. We write $\mathbb{Q}_p(n)$ for the 1-dimensional space over $\mathbb{Q}_p$ on which $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by ε_p^n and similarly for $\mathbb{Z}_p(n)$. We denote the residual representation of ε_p by ω_p. We will drop the p when it is clear from the context.

Let Σ be a set of primes. For an L-function we write L^Σ to denote the restricted Euler product of L over primes not in Σ and L_Σ to denote the restricted Euler product over primes in Σ.

For a ring R, we let $\mathrm{M}_n(R)$ denote the set of n by n matrices with entries in R. For a matrix $x \in \mathrm{M}_{2n}(R)$, we write

$$x = \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix}$$

where a_x, b_x, c_x, and d_x are all in $\mathrm{M}_n(R)$. We drop the subscript x when it is clear from the context.

Denote the group $\mathrm{SL}_2(\mathbb{Z})$ by Γ_1. We refer to a subgroup of Γ_1 as a congruence subgroup if it contains $\Gamma(N)$ for some positive integer N. We denote the complex upper half-plane by $\mathfrak{h}^1$. As usual, $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathfrak{h}^1 \cup \mathbb{R} \cup \{\infty\}$ via linear fractional transformations. We let $\Gamma_1^{\mathrm{J}} = \Gamma_1 \ltimes \mathbb{Z}^2$ be the full Jacobi modular group, as defined in [9]. Recall that the symplectic group is defined by

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \{\gamma \in \mathrm{M}_{2n}(\mathbb{R}) : {}^t\gamma \iota_n \gamma = \iota_n\}, \quad \iota_n = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$$

where we write 1_n to denote the n by n identity matrix. We denote $\mathrm{Sp}_{2n}(\mathbb{Z})$ by Γ_n. Siegel upper half-space is given by

$$\mathfrak{h}^n = \{Z \in \mathrm{M}_n(\mathbb{C}) : {}^tZ = Z, \, \mathrm{Im}(Z) > 0\}.$$

Siegel upper half-space comes equipped with an action of $\mathrm{Sp}_{2n}(\mathbb{R})$ given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.$$

For a congruence subgroup $\Gamma \subseteq \Gamma_1$, we write $M_k(\Gamma)$ to denote the space of modular forms of weight k on the congruence subgroup Γ. For $f \in M_k(\Gamma)$, we denote the n^{th} Fourier coefficient of f by $a_f(n)$. Given a ring $R \subseteq \mathbb{C}$, we write $M_k(\Gamma, R)$ to denote the space of modular forms with Fourier coefficients in R. Let $S_k(\Gamma)$ denote the space of cusp forms. For $f_1, f_2 \in M_k(\Gamma)$ with f_1 or f_2 a cusp form, the Petersson product is given by

$$\langle f_1, f_2 \rangle = \frac{1}{[\overline{\Gamma}_1 : \overline{\Gamma}]} \int_{\Gamma \backslash \mathfrak{h}^1} f_1(z)\overline{f_2(z)} y^{k-2} dx\, dy$$

where $\overline{\Gamma}_1$ means $\Gamma_1 / \pm 1_2$ and $\overline{\Gamma}$ is the image of Γ in $\overline{\Gamma}_1$. We write $\mathbb{T}_R(\Gamma)$ for the usual Hecke algebra over the ring R for the congruence subgroup Γ. We drop Γ from the notation when it is clear from the context. We say f is a newform if it is an eigenform for all the Hecke operators $T(n)$ with Fourier expansion normalized so that the Fourier coefficients are equal to the eigenvalues. We write $S_k^{\text{new}}(\Gamma)$ to denote the space of newforms.

The only half-integral weight modular forms we will be interested in are the ones in Kohnen's $+$-space defined by

$$S^+_{k-1/2}(\Gamma_0(4)) = \{g \in S_{k-1/2}(\Gamma_0(4)) : a_g(n) = 0 \text{ if } (-1)^{k-1} n \equiv 2, 3 (\operatorname{mod} 4)\}.$$

The Petersson product on $S^+_{k-1/2}(\Gamma_0(4))$ is given by

$$\langle g_1, g_2 \rangle = \int_{\Gamma_0(4) \backslash \mathfrak{h}^1} g_1(z)\overline{g_2(z)} y^{k-5/2} dx\, dy.$$

We denote the space of Jacobi cusp forms on Γ_1^{J} by $J_{k,1}^{\text{cusp}}(\Gamma_1^{\mathrm{J}})$. The inner product is given by

$$\langle \phi_1, \phi_2 \rangle = \int_{\Gamma_1^{\mathrm{J}} \backslash \mathfrak{h}^1 \times \mathbb{C}} \phi_1(\tau, z)\overline{\phi_2(\tau, z)} v^{k-3} e^{-4\pi y^2/v} dx\, dy\, du\, dv$$

for $\phi_1, \phi_2 \in J_{k,1}^{\text{cusp}}(\Gamma_1^{\mathrm{J}})$ and $\tau = u + iv$, $z = x + iy$.

Given a congruence group $\Gamma \subseteq \mathrm{Sp}_{2n}(\mathbb{Z})$, we denote the space of Siegel modular forms of weight k for Γ by $\mathcal{M}_k(\Gamma)$. The space of cusp forms is denoted by $\mathcal{S}_k(\Gamma)$. For $\gamma \in \mathrm{Sp}^+_{2n}(\mathbb{R})$, the slash operator of γ on a Siegel modular form F of weight k is given by $(F|_k\gamma)(Z) = \det(C_\gamma Z + D_\gamma)^{-k} F(\gamma Z)$. For F and G two Siegel modular forms with at least one of them a cusp form for Γ of weight k, define the Petersson product of F and G by

$$\langle F, G \rangle = \frac{1}{[\overline{\Gamma}_n : \overline{\Gamma}]} \int_{\Gamma \backslash \mathfrak{h}^n} F(Z)\overline{G(Z)} \det(Y)^k d\mu(Z).$$

We write $\mathbb{T}_{S,R}(\Gamma)$ for the usual Hecke algebra generated over R by the Hecke operators on Siegel modular forms for the congruence group Γ. We drop Γ from the notation when it is clear from the context. For a thorough treatment of Hecke operators on Siegel modular forms one can consult [1].

We will mainly be interested in the case when $F \in \mathcal{S}_k(\Gamma_2)$. Let $F \in \mathcal{S}_k(\Gamma_2)$ be a Hecke eigenform with eigenvalues $\lambda_F(m)$. The standard zeta function associated to F is given by

$$L_{\mathrm{st}}(s,F) = \prod_{\ell} W_\ell(\ell^{-s})^{-1} \tag{1}$$

where

$$W_\ell(t) = (1-\ell^2 t)\prod_{i=1}^{2}(1-\ell^2\alpha_{\ell,i}t)(1-\ell^2\alpha_{\ell,i}^{-1}t)$$

with $\alpha_{\ell,i}$ denoting the Satake parameters. Given a Hecke character ϕ, the twisted standard zeta function is given by

$$L_{\mathrm{st}}(s,F,\phi) = \prod_{\ell} W_\ell(\phi(\ell)\ell^{-s})^{-1}.$$

Associated to F is another L-function called the Spinor L-function. It is defined by

$$L_{\mathrm{spin}}(s,F) = \zeta(2s-2k+4)\sum_{m=1}^{\infty}\lambda_F(m)m^{-s}.$$

We will also be interested in the Maass space $\mathcal{M}_k^*(\Gamma_2) \subset \mathcal{M}_k(\Gamma_2)$. A Siegel modular form F is in the Maass space if the Fourier coefficients of F satisfy the relation

$$A_F(n,r,m) = \sum_{d\mid\gcd(n,r,m)} d^{k-1}A_F\left(\frac{nm}{d^2},\frac{r}{d},1\right)$$

for every $m,n,r \in \mathbb{Z}$ with $m,n,4mn-r^2 \geq 0$ ([53]).

3. The Saito-Kurokawa correspondence

In this section we review the explicit formula approach to the Saito-Kurokawa correspondence established by Maass ([26] - [28]), Andrianov [2], and Zagier [53]. We do not claim a complete account and are mainly concerned with stating the relevant facts we need in this paper. The interested reader is urged to consult the references for the details.

3.1. **The correspondence.**

The first step in establishing the Saito-Kurokawa correspondence is to relate the integer weight cusp forms of weight $2k-2$ and level 1 to half-integer weight modular forms of weight $k-1/2$ and level 4. This is accomplished via the Shimura and Shintani liftings. These maps are adjoint on cusp forms with respect to the Petersson products. Let D be a fundamental discriminant with $(-1)^{k-1}D > 0$. The Shimura lifting ζ_D is a map from $S_{k-1/2}^{+}(\Gamma_0(4))$ to $S_{2k-2}(\Gamma_1)$. Explicitly, for

$$g(z) = \sum c_g(n)q^n \in S_{k-1/2}^{+}(\Gamma_0(4M))$$

one has

$$\zeta_D g(z) = \sum_{n=1}^{\infty}\left(\sum_{d\mid n}\left(\frac{D}{d}\right)d^{k-2}c_g(|D|n^2/d^2)\right)q^n$$

where the summation defining $g(z)$ is over all $n \geq 1$ so that $(-1)^{k-1}n \equiv 0,1 (\bmod 4)$. On the other hand, the Shintani lifting ζ_D^* is a map from $S_{2k-2}(\Gamma_1)$ to $S_{k-1/2}^{+}(\Gamma_0(4))$. One can consult [23] for a precise defintion of the Shintani map as its precise definition will not be needed here. Using these liftings, one has the following theorem:

Theorem 3.1. *([22]) For D a fundamental discriminant with $(-1)^{k-1}D > 0$, the Shimura and Shintani liftings give Hecke-equivariant isomorphisms between $S_{2k-2}(\Gamma_1)$ and $S^+_{k-1/2}(\Gamma_0(4))$.*

Let $\mathcal{O}$ be a ring so that an embedding of $\mathcal{O}$ into $\mathbb{C}$ exists. Choose such an embedding and identify $\mathcal{O}$ with its image in $\mathbb{C}$ via this embedding. Assume that $\mathcal{O}$ contains all the Fourier coefficients of f. The Shintani lifting $g_f := \zeta_D^* f$ is determined only up to normalization by a constant multiple. However, we do have the following result of Stevens.

Theorem 3.2. *([47], Prop. 2.3.1) Let $f \in S_{2k-2}(\Gamma_1)$ be a newform. If the Fourier coefficients of f are in $\mathcal{O}$ then there exists a corresponding Shintani lifting g_f of f with Fourier coefficients in $\mathcal{O}$ as well.*

Remark 3.3. Throughout this paper we fix our g_f to have Fourier coefficients in $\mathcal{O}$ as in Theorem 3.2. If, in addition, $\mathcal{O}$ is a discrete valuation ring, we fix our g_f to have Fourier coefficients in $\mathcal{O}$ with some Fourier coefficient in $\mathcal{O}^\times$.

We have the following theorem relating half-integral weight cusp forms to Jacobi forms.

Theorem 3.4. *([9], Theorem 5.4) The map defined by*

$$\sum_{\substack{D<0,\, r\in\mathbb{Z}\\ D\equiv r^2 (\mathrm{mod}\, 4)}} c(D,r)e\left(\frac{r^2-D}{4}\tau + rz\right) \mapsto \sum_{\substack{D<0\\ D\equiv 0,1 (\mathrm{mod}\, 4)}} c(D)e(|D|\tau)$$

is a canonical Hecke equivariant isomorphism between $J^{\mathrm{cusp}}_{k,1}(\Gamma_1^{\mathrm{J}})$ and $S^+_{k-1/2}(\Gamma_0(4))$ preserving the Hilbert space structures.

Our final step is to relate Jacobi forms to Siegel forms. Let $F \in \mathcal{S}_k^*(\Gamma_2)$. One has that F admits a Fourier-Jacobi expansion

$$F(\tau, z, \tau') = \sum_{m\geq 0} \phi_m(\tau, z)e(m\tau')$$

where the ϕ_m are Jacobi forms of weight k, index m, and level 1.

Theorem 3.5. *([9], Theorem 6.2) The association $F \mapsto \phi_1$ gives a Hecke equivariant isomorphism between $\mathcal{S}_k^*(\Gamma_2)$ and $J^{\mathrm{cusp}}_{k,1}(\Gamma_1^{\mathrm{J}})$. The inverse map is given by sending $\phi(\tau,z) \in J^{\mathrm{cusp}}_{k,1}(\Gamma_1^{\mathrm{J}})$ to $F(\tau,z,\tau') = \sum_{m\geq 0} V_m\phi(\tau,z)e(m\tau')$ where V_m is the index shifting operator as defined in ([9], Section 4).*

Corollary 3.6. *Let $\phi \in J^{\mathrm{cusp}}_{k,1}(\Gamma_1^{\mathrm{J}}, \mathcal{O})$ where $\mathcal{O}$ is some ring. If F is the Siegel modular form associated to ϕ in Theorem 3.5 then F has Fourier coefficients in $\mathcal{O}$.*

Proof. Using that F is in the Maass space and the definition of V_m we obtain

$$A(n,r,m) = \sum_{d|\gcd(m,n,r)} d^{k-1} c\left(\frac{4nm-r^2}{d^2}, \frac{r}{d}\right)$$

where the $c(D,r)$ are the Fourier coefficients of ϕ. The rest is clear. $\square$

Combining these results one obtains the Saito-Kurokawa correspondence.

Theorem 3.7. *([53]) The space $\mathcal{S}_k^*(\Gamma_2)$ is spanned by Hecke eigenforms. These are in 1-1 correspondence with newforms $f \in S_{2k-2}(\Gamma_1)$, the correspondence being such that if F_f correponds to f, then one has*

$$L_{\text{spin}}(s, F_f) = \zeta(s-k+1)\zeta(s-k+2)L(s,f). \tag{2}$$

Corollary 3.8. *The Saito-Kurokawa isomorphism is a Hecke-equivariant isomorphism over $\mathcal{O}$. In particular, if $\mathcal{O}$ is a discrete valuation ring, F_f has a Fourier coefficient in $\mathcal{O}^\times$.*

We also note the following theorem giving an equation relating $\langle F_f, F_f\rangle$ to $\langle f, f\rangle$.

Theorem 3.9. *([24], [25]) Let $f \in S_{2k-2}(\Gamma_1)$ be a newform, $F_f \in \mathcal{S}_k^*(\Gamma_2)$ the corresponding Saito-Kurokawa lift, and $g(z) = \sum c_g(n)q^n$ the weight $k-1/2$ cusp form corresponding to f under the Shintani map. We have the following inner product relation*

$$\langle F_f, F_f\rangle = \frac{(k-1)}{2^5 3^2 \pi} \cdot \frac{c_g(|D|)^2}{|D|^{k-3/2}} \cdot \frac{L(k,f)}{L(k-1,f,\chi_D)} \langle f, f\rangle$$

where D is a fundamental discriminant so that $(-1)^{k-1}D > 0$ and χ_D is the quadratic character associated to D.

The standard zeta function of F_f can be factored into a particularly simple form, as given in the following theorem.

Theorem 3.10. *Let N be a positive integer, Σ the set of primes dividing N, and χ a Dirichlet character of conductor N. Let $f \in S_{2k-2}(\Gamma_1)$ be a newform and F_f the corresponding Saito-Kurokawa lift of f. The standard zeta function of F_f factors as*

$$L_{\text{st}}^\Sigma(2s, F_f, \chi) = L^\Sigma(2s-2, \chi)L^\Sigma(2s+k-3, f, \chi)L^\Sigma(2s+k-4, f, \chi).$$

Proof. To prove this theorem we need to relate the Satake parameters $\alpha_i := \alpha_{p,i}$ to the eigenvalues of f in order to decompose the standard zeta function. To accomplish this, we use the following formula (see [29]):

$$L_{\text{spin},(p)}(s, F_f) = (1-\alpha_0 p^{-s})(1-\alpha_0\alpha_1 p^{-s})(1-\alpha_0\alpha_2 p^{-s})(1-\alpha_0\alpha_1\alpha_2 p^{-s}).$$

Recall that by Equation 2 we have

$$L_{\text{spin},(p)}(s, F_f) = (1-p^{k-1-s})(1-p^{k-s-2})(1-a_f(p)p^{-s}+p^{2k-3-2s}).$$

Letting $x = p^{-s}$, we have the polynomial identity

$$(1-\alpha_0 x)(1-\alpha_0\alpha_1 x)(1-\alpha_0\alpha_2 x)(1-\alpha_0\alpha_1\alpha_2 x) = (1-p^{k-1}x)(1-p^{k-2}x)(1-a_f(p)x+p^{2k-3}x^2).$$

Therefore we have that $\{\alpha_0, \alpha_0\alpha_1, \alpha_0\alpha_2, \alpha_0\alpha_1\alpha_2\} = \left\{p^{k-1}, p^{k-2}, \frac{2p^{2k-3}}{a_f(p)\pm\sqrt{a_f(p)^2-4p^{2k-3}}}\right\}$. The values $\alpha_0\alpha_1$ and $\alpha_0\alpha_2$ are completely symmetrical so we set

$$\alpha_0\alpha_1 = \frac{2p^{2k-3}}{a_f(p)+\sqrt{a_f(p)^2-4p^{2k-3}}}$$

and

$$\alpha_0\alpha_2 = \frac{2p^{2k-3}}{a_f(p)-\sqrt{a_f(p)^2-4p^{2k-3}}}.$$

Since we have $\alpha_0^2\alpha_1\alpha_2 = p^{2k-3}$, $\alpha_0 = p^{k-1}$ or p^{k-2} but is arbitrary up to this choice. We fix $\alpha_0 = p^{k-1}$. Pick α_p and β_p such that

$$\alpha_p + \beta_p = a_f(p)$$

and

$$\alpha_p\beta_p = p^{2k-3}.$$

Thus,

$$\alpha_1 = \beta_p p^{1-k}$$

and

$$\alpha_2 = \alpha_p p^{1-k}.$$

Therefore we can write

$$(1-\chi(p)\alpha_1 p^{2-2s})(1-\chi(p)\alpha_2 p^{2-2s}) = 1-\chi(p)a_f(p)p^{3-2s-k} + \chi(p)^2 p^{3-4s}$$

and

$$(1-\chi(p)\alpha_1^{-1} p^{2-2s})(1-\chi(p)\alpha_2^{-1} p^{2-2s}) = 1-\chi(p)a_f(p)p^{4-2s-k} + \chi(p)^2 p^{5-4s}.$$

Substituting this back in for $L^{\Sigma}(2s, F_f, \chi)$ we have the result. $\square$

4. Eisenstein series

In this section we study an Eisenstein series $E(Z, s, \chi)$ as defined by Shimura ([39], [42], [44]). We begin with basic definitions and then move to a study of the Fourier coefficients of the Eisenstein series. We show that under a suitable normalization for a certain value of s we have that $E(Z, s, \chi)$ is a holomorphic Siegel modular form with Fourier coefficients that are p-integral for a prime p of our choosing. We next move to studying an inner product relation of Shimura that calculates the inner product of $E(Z, s, \chi)$ with a Siegel cusp form F in terms of F and the standard zeta function associated to F.

4.1. **Basic definitions.** Before we can define the Eisenstein series we need to define some subgroups of $\mathrm{Sp}_{2n}(\mathbb{A})$ and $\mathrm{Sp}_{2n}(\mathbb{Q})$. Let $\mathfrak{a}$ and $\mathfrak{b}$ be non-zero ideals in $\mathbb{Z}$. Set

$$D[\mathfrak{a},\mathfrak{b}] = \mathrm{Sp}_{2n}(\mathbb{R}) \prod_{\ell \in \mathbf{f}} D_\ell[\mathfrak{a},\mathfrak{b}]$$

where

$$D_\ell[\mathfrak{a},\mathfrak{b}] = \{x \in \mathrm{Sp}_{2n}(\mathbb{Q}_\ell) : a_x \in \mathrm{M}_n(\mathbb{Z}_\ell), b_x \in \mathrm{M}_n(\mathfrak{a}_\ell), c_x \in \mathrm{M}_n(\mathfrak{b}_\ell), d_x \in \mathrm{M}_n(\mathbb{Z}_\ell)\}.$$

Define a maximal compact subgroup C_υ of $\mathrm{Sp}_{2n}(\mathbb{Q}_\upsilon)$ by

$$C_\upsilon = \begin{cases} \{\alpha \in \mathrm{Sp}_{2n}(\mathbb{R}) : \alpha(i) = i\} & \upsilon = \infty, \\ \mathrm{Sp}_{2n}(\mathbb{Q}_\upsilon) \cap \mathrm{GL}_{2n}(\mathbb{Z}_\upsilon) & \upsilon \in \mathbf{f}, \end{cases}$$

and set $C = \prod C_\upsilon$. It is understood here that i denotes the $n \times n$ identity matrix multiplied by the complex number i. Let P be the Siegel parabolic of $\mathrm{Sp}_{2n}(\mathbb{Q})$ defined by

$$P = \{x \in \mathrm{Sp}_{2n}(\mathbb{Q}) : c_x = 0\}.$$

Set

$$\mathbb{S}^n(R) = \{x \in \mathrm{M}_n(R) : {}^t x = x\}.$$

We write elements $Z \in \mathfrak{h}^n$ as $Z = X + iY$ with $X, Y \in \mathbb{S}^n(\mathbb{R})$ and $Y > 0$.

Let $\lambda = \frac{n+1}{2}$, N a positive integer, Σ the set of primes dividing N and k a positive integer such that $k > \max\{3, 2\lambda\}$. In order to define the Eisenstein series we need a Hecke character χ of $\mathbb{A}^\times$ satisfying

$$\begin{aligned} \chi_\infty(x) &= \operatorname{sgn}(x)^k, \\ \chi_\ell(a) &= 1 \quad \text{if } \ell \in \mathbf{f},\ a \in \mathbb{Z}_\ell^\times, \text{ and } N \mid (a-1). \end{aligned} \tag{3}$$

Set $D = D[1, N]$ and define functions μ and ε on $\operatorname{Sp}_{2n}(\mathbb{A})$ by

$$\begin{aligned} \mu(x) &= 0 \quad \text{if } x \notin P(\mathbb{A})D, \\ \mu(pw) &= \chi(\det(d_p))^{-1}\chi_\Sigma(\det(d_w))^{-1}\det(d_p)^{-k} \quad \text{if } x = pw \in P(\mathbb{A})D, \end{aligned}$$

and

$$\begin{aligned} \varepsilon(x_\infty) &= |j(x_\infty, i)|^2 \\ \varepsilon(x_\mathbf{f}) &= \det(d_p)^{-2} \quad \text{for } x = pw \end{aligned}$$

where $\chi_\Sigma = \prod_{\ell \in \Sigma} \chi_\ell$ and $j(x_\infty, Z) = \det(c_{x_\infty} Z + d_{x_\infty})$.

We now have all the ingredients necessary to define the Eisenstein series we are interested in. For $x \in \operatorname{Sp}_{2n}(\mathbb{A})$ and $s \in \mathbb{C}$, define

$$E(x, s) = E(x, s; \chi, D) = \sum_{\alpha \in A} \mu(\alpha x)\varepsilon(\alpha x)^{-s}, \quad A = P \backslash \operatorname{Sp}_{2n}(\mathbb{Q}).$$

This gives us an Eisenstein series defined on $\operatorname{Sp}_{2n}(\mathbb{A}) \times \mathbb{C}$, but we will ultimately be interested in an Eisenstein series $E(Z, s)$ defined on $\mathfrak{h}^n \times \mathbb{C}$. The Eisenstein series $E(Z, s)$ converges locally uniformly in $\mathfrak{h}^n$ for $\operatorname{Re}(s) > \lambda$. We associate the Eisenstein series $E(Z, s)$ to $E(x, s)$ as follows.

More generally, let F_0 be a function on $\operatorname{Sp}_{2n}(\mathbb{A})$ such that

$$F_0(\alpha x w) = F_0(x)J(w, i)^{-1} \quad \text{for } \alpha \in \operatorname{Sp}_{2n}(\mathbb{Q}) \text{ and } w \in C' \tag{4}$$

where C' is an open subgroup of C and $J(x, z)$ is defined by

$$J(x, z) = J_{k,s}(x, z) = j(x, z)^k |j(x, z)|^s.$$

Our Eisenstein series is such a function. Let $\Gamma' = \operatorname{Sp}_{2n}(\mathbb{Q}) \cap \operatorname{Sp}_{2n}(\mathbb{R})C'$ and define a function F on $\mathfrak{h}^n$ by

$$F(x(i)) = F_0(x)J(x, i) \quad \text{for } x \in \operatorname{Sp}_{2n}(\mathbb{R})C'. \tag{5}$$

Using the strong approximation theorem ($\operatorname{Sp}_{2n}(\mathbb{A}) = \operatorname{Sp}_{2n}(\mathbb{Q})\operatorname{Sp}_{2n}(\mathbb{R})C'$) we have that F is well-defined and satisfies

$$F(\gamma Z) = F(Z)J(\gamma, Z) \quad \text{for } \gamma \in \Gamma' \text{ and } Z \in \mathfrak{h}^n. \tag{6}$$

Therefore, we have an associated Eisenstein series $E(Z, s)$ defined on $\mathfrak{h}^n \times \mathbb{C}$. The Eisenstein series $E(Z, s)$ converges locally uniformly in $\mathfrak{h}^n$ for $\operatorname{Re}(s) > \lambda$. Conversely, given a function F satisfying Equation 6, we can define a function F_0 satisfying Equation 4 and Equation 5 by

$$F_0(\alpha x) = F(x(i))J(x, i)^{-1} \quad \text{for } \alpha \in \operatorname{Sp}_{2n}(\mathbb{Q}) \text{ and } x \in \operatorname{Sp}_{2n}(\mathbb{R})C'.$$

We will also make use of the fact that if $G = F|_{\gamma^{-1}}$ for $\gamma \in \Gamma_n$ with F a Siegel modular form, then $G_0(x) = F_0(x\gamma_\mathbf{f})$ and vice versa.

4.2. **The Fourier coefficients of $E(Z,s,\chi)$.** We will now focus our attention on the Fourier coefficients of $E(x,s)$ and in turn $E(Z,s)$. It turns out that it is easier to study the Fourier coefficients of a simple translation of $E(x,s)$ given by

$$E^*(x,s) = E(x\iota_{\mathbf{f}}^{-1}, s; \chi, D)$$

where we recall $\iota = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}$ ([39]). Using the discussion above, we get a corresponding form $E^*(Z,s)$.

Let $L = \mathbb{S}^n(\mathbb{Q}) \cap \mathrm{M}_n(\mathbb{Z})$, $L' = \{\mathfrak{s} \in \mathbb{S}^n(\mathbb{Q}) : \mathrm{Tr}(\mathfrak{s}L) \subseteq \mathbb{Z}\}$ and $M = N^{-1}L'$. The Eisenstein series $E^*(Z,s)$ has a Fourier expansion

$$E^*(Z,s) = \sum_{h \in M} a(h,Y,s) e(\mathrm{Tr}(hX))$$

for $Z = X + iY \in \mathfrak{h}^n$ ([39]).

Remark 4.1. The Fourier coefficients of $E^*(Z,s)$ are nonvanishing only when h is totally positive definite due to the fact that we have restricted our k to be larger then 3 ([39], Page 460).

We have the following result of Shimura explicitly calculating the Fourier coefficients $a(h,Y,s)$.

Proposition 4.2. *([44], Prop. 18.7, 18.14) For $N \neq 1$,*

$$\begin{aligned} a(h,Y,s) = {} & \det(Y)^{-k/2} N^{-n\lambda} \det(Y)^s \alpha_N({}^t\overline{(Y^{1/2})} h Y^{1/2}; 2s, \chi) \\ & \cdot \xi(Y,h,s+k/2,s-k/2) \end{aligned}$$

where ξ is defined by

$$\xi(Y,h;s,t) = \int_{\mathbb{S}^n(\mathbb{R})} e(-\mathrm{Tr}(hX)) \det(X+iY)^{-s} \det(X-iY)^{-t} dX$$

with $0 < Y \in \mathbb{S}^n(\mathbb{R})$, $h \in \mathbb{S}^n(\mathbb{R})$, $s,t \in \mathbb{C}$ and α_N is a Whittaker integral. One can consult [44] for the definition of α_N; it will not be needed here.

For a Dirichlet character ψ, set $\Lambda^{\Sigma}(s,\psi) = L^{\Sigma}(2s,\psi) \prod_{j=1}^{[n/2]} L^{\Sigma}(4s-2j,\psi^2)$. We normalize $E^*(Z,s)$ by multiplying it by $\pi^{-\frac{n(n+2)}{4}} \Lambda^{\Sigma}(s,\chi)$ and call this normalized Eisenstein series $D_{E^*}(Z,s) = D_{E^*}(Z,s;k,\chi,N)$. Consider the Fourier expansion of $D_{E^*}(Z,s)$ at $s = \lambda - k/2$:

$$\begin{aligned} D_{E^*}(Z,\lambda-k/2) &= \sum_{h \in M} \pi^{-\frac{n(n+2)}{4}} \Lambda^{\Sigma}(\lambda-k/2,\chi) a(h,Y,\lambda-k/2) e(\mathrm{Tr}(hX)) \\ &= \sum_{h \in M} b(h,Y,\lambda-k/2) e(\mathrm{Tr}(hX)). \end{aligned}$$

The normalized Eisenstein series $D_{E^*}(Z,\lambda-k/2)$ is in $\mathcal{M}_k(\mathbb{Q}^{\mathrm{ab}})$ where $\mathbb{Q}^{\mathrm{ab}}$ is the maximal abelian extension of $\mathbb{Q}$ ([41], Prop. 4.1). We show that the coefficients of $D_{E^*}(Z,\lambda-k/2)$ actually lie in a finite extension of $\mathbb{Z}_p$ for a suitably chosen prime p. Using ([37], 4.34K, 4.35IV) we have that

$$\xi(Y,h;\lambda,\lambda-k) = \frac{i^{nk} \pi^{\frac{n(n+2)}{4}} 2^{n(k-1)} \det(Y)^{k-\lambda}}{\mathcal{P}_n} e(i\ \mathrm{Tr}(hY)), \tag{7}$$

where

$$\mathcal{P}_n = \prod_{j=0}^{[\lambda]} j! \prod_{j=0}^{[\lambda]-1} \frac{(2j+1)!!}{2^{j+1}}$$

and

$$n!! = \begin{cases} n(n-2)\dots 5\cdot 3\cdot 1 & n>0, \text{ odd} \\ n(n-2)\dots 6\cdot 4\cdot 2 & n>0, \text{ even.} \end{cases}$$

Using that h is totally positive definite we have:

Proposition 4.3. *([44], Prop. 19.2) Set χ_h to be the Hecke character corresponding to $\mathbb{Q}(\sqrt{-\det(h)})/\mathbb{Q}$. Then*

$$\alpha_N(h,s,\chi) = \Lambda^{\Sigma}(s,\chi)^{-1}\Lambda_h^{\Sigma}(s,\chi)\prod_{\ell\in\mathcal{C}} f_{h,Y,\ell}(\chi(\ell)|\ell|^{2s})$$

where $\mathcal{C}$ is a finite subset of $\boldsymbol{f}$, the $f_{h,Y,\ell}$ are polynomials with a constant term of 1 and coefficients in $\mathbb{Z}$ independent of χ, and

$$\Lambda_h^{\Sigma}(s,\chi) = \begin{cases} L^{\Sigma}(2s-n/2,\chi\chi_h) & n\in 2\mathbb{Z} \\ 1 & \textit{otherwise.} \end{cases}$$

To ease the notation set $\mathcal{F}_{h,Y}(s,\chi) = \prod_{\ell\in\mathcal{C}} f_{h,Y,\ell}(\chi(\ell)|\ell|^{s})$. Combining Equation 7, Corollary 4.2, and Proposition 4.3 we have

$$b(h,Y,\lambda-k/2) = \begin{cases} \frac{i^{nk}2^{n(k-1)}L^{\Sigma}(2\lambda-k-n/2,\chi\chi_h)\mathcal{F}_{Y,h}(2\lambda-k,\chi)}{N^{n\lambda}\mathcal{P}_n} e(i\ \mathrm{Tr}(hY)) & n\in 2\mathbb{Z} \\ \frac{i^{nk}2^{n(k-1)}\mathcal{F}_{Y,h}(2\lambda-k,\chi)}{N^{n\lambda}\mathcal{P}_n} e(i\ \mathrm{Tr}(hY)) & \text{otherwise.} \end{cases}$$

Let p be an odd prime with $\gcd(p,N)=1$ and $p>2\lambda-1$. We show that the $b(h,Y,\lambda-k/2)$ all lie in $\mathbb{Z}_p[\chi,i^{nk}]$ where $\mathbb{Z}_p[\chi]$ is the extension of $\mathbb{Z}_p$ generated by the values of χ. It is clear that $i^{nk}2^{n(k-1)}N^{-n\lambda}\in\mathbb{Z}_p[\chi,i^{nk}]$ by our choice of p. The fact that $p>2\lambda-1$ and $n\geq 1$ so that $2\lambda-1\geq\lambda$ shows that $\mathcal{P}_n$ is in $\mathbb{Z}_p$. The fact that we have chosen $k>2\lambda$ gives us that $2\lambda-k<0$. This in turn shows that $|p|^{2\lambda-k}=p^{k-2\lambda}\in\mathbb{Z}_p$. Using this fact and that the coefficients of $f_{h,Y,\ell}$ all lie in $\mathbb{Z}$, we have that $\mathcal{F}_{Y,h}(2\lambda-k,\chi)\in\mathbb{Z}_p[\chi,i^{nk}]$ for all h. Therefore it remains to show that $L^{\Sigma}(2\lambda-k-n/2,\chi\chi_h)\in\mathbb{Z}_p[\chi,i^{nk}]$. We will in fact show that for any Dirichlet character ψ of conductor N and any positive integer n that $L^{\Sigma}(1-n,\psi)\in\mathbb{Z}_p[\psi]$.

Let $\omega:\mathbb{Z}_p^{\times}\to\mu_{p-1}$ be the usual Teichmuller character. One has the existence of a p-adic L-function $\mathcal{L}_p(s,\chi)$ defined on $\{s\in\mathbb{C}_p : |s|<(p-1)p^{-1/(p-1)}\}$ such that

$$\mathcal{L}_p(1-n,\psi) = (1-\psi\omega^{-n}(p)p^{n-1})\,\frac{B_{n,\psi\omega^{-n}}}{n}$$

for $n\geq 1$ ([51], Theorem 5.11). Using this and the well-known fact that one has $L(1-n,\psi)=-\frac{B_{n,\psi}}{n}$ where $B_{n,\psi}$ is the generalized Bernoulli number defined by

$$\sum_{a=1}^{N}\frac{\psi(a)te^{at}}{e^{Nt}-1} = \sum_{j=0}^{\infty} B_{j,\psi}\frac{t^j}{j!},$$

we can write

$$L^{\Sigma}(1-n,\psi) = -(1-\psi(p)p^{n-1})^{-1}\prod_{\ell\mid N}(1-\psi(\ell)\ell^{1-n})\mathcal{L}_p(1-n,\psi\omega^{n}).$$

One can see that $(1-\psi(p)p^{n-1})^{-1} \in \mathbb{Z}_p[\psi]$ by expanding it in a convergent geometric series. We use the fact that $\gcd(p,N)=1$ to conclude that $\prod_{\ell \mid N}(1-\psi(\ell)\ell^{1-n})$ lies in $\mathbb{Z}_p[\psi]$. To finish our proof that $L^{\Sigma}(1-n,\psi) \in \mathbb{Z}_p[\psi]$ for all $n \in \mathbb{N}$, we note that $\mathcal{L}_p(m,\psi)$ is a p-adic integer for all m and all ψ with conductor N such that $\gcd(p,N)=1$ by ([51], Corl. 5.13). Therefore we have proven:

Theorem 4.4. *Let n, N, and k be positive integers such that $k > \max\{3, n+1\}$. Let χ be a Dirichlet character as in Equation 3. Let p be an odd prime such that $p > n$ and $(p,N)=1$. Then $D_{E^*}(Z,(n+1)/2-k/2)$ is in $\mathcal{M}_k(\Gamma_0^n(N), \mathbb{Z}_p[\chi, i^{nk}])$ for*

$$\Gamma_0^n(N) = \{\gamma \in \Gamma_n : c_\gamma \equiv 0 (\mathrm{mod}\, N)\}.$$

4.3. **Pullbacks and an inner product relation.** In this section we will use the results in the previous section specialized to the case $n=4$.

We turn our attention to studying the pullback of the Eisenstein series $E(\mathfrak{z}, s, \chi)$ via maps

$$\begin{array}{ccc} \mathfrak{h}^2 \times \mathfrak{h}^2 & \hookrightarrow & \mathfrak{h}^4 \\ (Z,W) & \mapsto & \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} = \mathrm{diag}[Z,W] \end{array}$$

and

$$\begin{array}{ccc} \Gamma_2 \times \Gamma_2 & \hookrightarrow & \Gamma_4 \\ (\alpha,\beta) & \mapsto & \alpha \times \beta = \begin{pmatrix} a_\alpha & 0 & b_\alpha & 0 \\ 0 & a_\beta & 0 & b_\beta \\ c_\alpha & 0 & d_\alpha & 0 \\ 0 & c_\beta & 0 & d_\beta \end{pmatrix}. \end{array}$$

These pullbacks have been studied extensively by Shimura ([43], [44]) as well as by Garrett ([15], [16]). In particular, if one has a Siegel modular form G on Γ_4 of weight k and level N, then its pullback to $\Gamma_2 \times \Gamma_2$ is a Siegel modular form in each of the variables Z and W of weight k and level N. We will be interested primarily in the results found in [43], particularly the inner product relation found there.

Let $\sigma_{\mathbf{f}} \in \mathrm{Sp}_8(\mathbb{Q}_{\mathbf{f}})$ be defined as $\sigma_{\mathbf{f}} = (\sigma_\ell)$ with

$$\sigma_\ell = \begin{cases} I_8 & \text{if } \ell \nmid N \\ \begin{pmatrix} I_4 & 0_4 \\ \begin{pmatrix} 0_2 & I_2 \\ I_2 & 0_2 \end{pmatrix} & I_4 \end{pmatrix} & \text{if } \ell \mid N. \end{cases}$$

The strong approximation gives an element $\rho \in \Gamma_4 \cap D[1,N]\sigma_{\mathbf{f}}$ such that $N_\ell \mid a(\sigma_{\mathbf{f}}\rho^{-1})_\ell - I_4$ for every $\ell \mid N$. In particular, we have that $E|_\rho$ corresponds to $E(x\sigma_{\mathbf{f}}^{-1})$.

Let $F \in \mathcal{S}_k(\Gamma_0^2(N), \mathbb{R})$ be a Siegel eigenform. We specialize a result of Shimura that gives the inner product of $E|_\rho$ with such an F. Applying ([43], Equation 6.17) to our situation we get

$$\langle D_{E|_\rho}(\mathrm{diag}[Z,W], (5-k)/2), (F|_\iota)^c(W)\rangle = \pi^{-3}\,\mathcal{A}_{k,N} L_{\mathrm{st}}^{\Sigma}(5-k,F,\chi)F(Z) \tag{8}$$

where $\mathcal{A}_{k,N} = \dfrac{(-1)^k\, 2^{2k-3} \upsilon_N}{3\,[\Gamma_2 : \Gamma_0^2(N)]}$, $\upsilon_N = \pm 1$, $L_{\mathrm{st}}^{\Sigma}(5-k,F,\chi)$ is the standard zeta function as defined in Equation 1, and $(F|_\iota)^c$ denotes taking the complex conjugates

of the Fourier coefficients of $F|_\iota$ where $F|_\iota$ is now a Siegel form on

$$\Gamma^{2,0}(N) = \left\{ \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \in \Gamma_2 | B_2 \equiv 0 (\bmod N) \right\}.$$

We can use the q-expansion principle for Siegel modular forms ([6], Prop. 1.5) to conclude that $F|_\iota$ has real Fourier coefficients since we chose F to have real Fourier coefficients. Therefore $(F|_\iota)^c(W)$ in Equation 8 becomes $(F|_\iota)(W)$. Thus we have

$$\langle D_{E|_{\rho(1\times\iota_2^{-1})}}(\mathrm{diag}[Z,W],(5-k)/2), F(W)\rangle = \pi^{-3}\,\mathcal{A}_{k,N} L^{\Sigma}_{\mathrm{st}}(5-k,F,\chi)F(Z).$$

Our next step is to make sure that the Fourier coefficients of $\mathcal{E}(Z,W)$ are still in some finite extension of $\mathbb{Z}_p$, where

$$\mathcal{E}(Z,W) := D_{E|_{\rho(1\times\iota_2^{-1})}}(\mathrm{diag}[Z,W],(5-k)/2).$$

Recall from Theorem 4.4 that $D_{E^*}(Z,(5-k)/2) \in \mathcal{M}_k(\Gamma_0^4(N), \mathbb{Z}_p[\chi])$. Therefore, applying the q-expansion principle ([6], Prop. 1.5) to $D_{E^*}(\mathrm{diag}[Z,W],(5-k)/2)$ slashed by $\iota_4^{-1}\rho(1\times\iota_2^{-1})$, we get that $D_{E|_{\rho(1\times\iota_2^{-1})}}(\mathrm{diag}[Z,W],(5-k)/2)$ has Fourier coefficients in $\mathbb{Z}_p[\chi]$.

Summarizing, we have the following theorem.

Theorem 4.5. *Let $N>1$ and $k>3$. For $F \in S_k(\Gamma_0^2(N),\mathbb{R})$ a Hecke eigenform and p a prime with $p>2$ and $\gcd(p,N)=1$ we have*

$$\langle \mathcal{E}(Z,W), F(W)\rangle = \pi^{-3}\,\mathcal{A}_{k,N} L^{\Sigma}_{\mathrm{st}}(5-k,F,\chi)F(Z) \tag{9}$$

with $\mathcal{E}(Z,W)$ having Fourier coefficients in $\mathbb{Z}_p[\chi]$.

5. Periods and a certain Hecke operator

Throughout this section we make the following assumptions. Let k be a positive integer with $k \geq 2$. Let p be a prime so that $p > 2k-2$. We let K be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}$ and uniformizer ϖ. Fix an embedding of K into $\mathbb{C}$ compatible with the embeddings fixed in Section 2. Let $\mathfrak{p}$ be the prime of $\mathcal{O}$ lying over p.

5.1. **Periods associated to newforms.** Let $f \in S_{2k-2}(\Gamma_1)$ be a newform with eigenvalues in $\mathcal{O}$. The congruence class of f modulo p is the set of eigenforms with eigenvalues congruent to those of f modulo p. The congruence class of f in $S_{2k-2}(\Gamma_1)$ determines a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_{\mathcal{O}}$ and a residual representation

$$\rho_{\mathfrak{m}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{T}_{\mathcal{O}}/\mathfrak{m}),$$

so that $\mathrm{Tr}(\rho_{\mathfrak{m}}(\mathrm{Frob}_\ell)) = T(\ell)$ for all primes $\ell \neq p$ where $\mathbb{T}_{\mathcal{O}}/\mathfrak{m}$ is of characteristic p. This fact is essentially due to Deligne, see ([32], Prop. 5.1) for a detailed proof.

Associated to f is a surjective $\mathcal{O}$-algebra map $\pi_f : \mathbb{T}_{\mathcal{O},\mathfrak{m}} \to \mathcal{O}$ given by $T(\ell) \mapsto a_f(\ell)$. We can view this as a map into $\mathbb{C}$ as well via the embeddings $\mathcal{O} \hookrightarrow K \hookrightarrow \mathbb{C}$ where the embedding of K into $\mathbb{C}$ was fixed at the beginning of this section. Let $\wp_f$ be the kernel of π_f.

For f so that $\rho_{\mathfrak{m}}$ is irreducible, one has complex periods $\Omega_f^{\pm}$ uniquely determined up to a $\mathcal{O}$-unit as defined in [50]. One should note that while Vatsal restricts to the case of level $N \geq 4$ in [50], one can also define the periods $\Omega_f^{\pm}$ for all levels by using the arguments given in ([19], Section 3). Using these periods we have the following theorem essentially due to Shimura.

Theorem 5.1. *([36], Theorem 1) Let $f \in S_{2k-2}(\Gamma_1, \mathcal{O})$ be a newform. There exist complex periods $\Omega_f^{\pm}$ such that for each integer m with $0 < m < 2k-2$ and every Dirichlet character χ one has*

$$\frac{L(m,f,\chi)}{\tau(\chi)(2\pi i)^m} \in \begin{cases} \Omega_f^{+}\mathcal{O}_\chi & \text{if} \quad \chi(-1) = (-1)^m \\ \Omega_f^{-}\mathcal{O}_\chi & \text{if} \quad \chi(-1) = (-1)^{m-1}, \end{cases}$$

where $\tau(\chi)$ is the Gauss sum of χ and $\mathcal{O}_\chi$ is the extension of $\mathcal{O}$ generated by the values of χ.

Using the periods $\Omega_f^{\pm}$ we make the following conjecture which we prove under the additional assumption that f is ordinary at p.

Conjecture 5.2. Let $f = f_1, f_2, \dots f_r$ be a basis of eigenforms for $S_{2k-2}(\Gamma_1)$ with f a newform. Enlarge $\mathcal{O}$ if necessary so that the basis is defined over $\mathcal{O}$. Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{T}_{\mathcal{O}}$ associated to f and assume that representation $\rho_{\mathfrak{m}}$ is irreducible. Then there exists a Hecke operator $t \in \mathbb{T}_{\mathcal{O}}$ so that

$$tf_i = \begin{cases} u\frac{\langle f,f\rangle}{\Omega_f^{+}\Omega_f^{-}}f & \text{if } i = 1 \\ 0 & \text{if } i \neq 1 \end{cases}$$

for u a unit in $\mathcal{O}$.

5.2. **A certain Hecke operator.** In this section we will establish the validity of Conjecture 5.2 in the case that f is ordinary at p.

Let $f = f_1, f_2, \dots, f_r$ be a basis of eigenforms for $S_{2k-2}(\Gamma_1)$ as in Conjecture 5.2. We enlarge K here if necessary so that this basis is defined over $\mathcal{O}$. As with f, there are maps π_{f_i} for each i as well as kernels $\wp_{f_i}$.

The fact that f is a newform allows us to write

$$\mathbb{T}_{\mathcal{O},\mathfrak{m}} \otimes_{\mathcal{O}} K = K \oplus D$$

for a K-algebra D so that π_f induces the projection of $\mathbb{T}_{\mathcal{O},\mathfrak{m}}$ onto K ([19]). In this direct sum, K corresponds to the Hecke algebra acting on the eigenspace generated by f and D corresponds to the Hecke algebra acting on the space generated by the rest of the f_i's. Let ϱ be the projection map of $\mathbb{T}_{\mathcal{O},\mathfrak{m}}$ to D. Set I_f to be the kernel of ϱ. Using that our Hecke algebra is reduced, it is clear from the definition that we have

$$I_f = \mathrm{Ann}(\wp_f) = \bigcap_{i=2}^{r} \wp_{f_i} \tag{10}$$

where $\mathrm{Ann}(\wp_f)$ denotes the annihilator of the ideal $\wp_f$. Since $\mathbb{T}_{\mathcal{O},\mathfrak{m}}$ is reduced, we have that $\wp_f \cap I_f = 0$. Therefore we have that

$$\mathbb{T}_{\mathcal{O},\mathfrak{m}}/(\wp_f \oplus I_f) = \mathbb{T}_{\mathcal{O},\mathfrak{m}}/(\wp_f, I_f) \xrightarrow{\simeq} \mathcal{O}/\pi_f(I_f)$$

where we use here that

$$\pi_f : \mathbb{T}_{\mathcal{O},\mathfrak{m}}/\wp_f \xrightarrow{\simeq} \mathcal{O}.$$

Since $\mathcal{O}$ is a principal ideal domain, there exists $a \in \mathcal{O}$ so that $\pi_f(I_f) = a\mathcal{O}$. Therefore we have

$$\mathcal{O}/a\mathcal{O} \cong \mathbb{T}_{\mathcal{O},\mathfrak{m}}/(\wp_f \oplus I_f). \tag{11}$$

For each prime ℓ, choose $\alpha_f(\ell)$ and $\beta_f(\ell)$ so that $\alpha_f(\ell)+\beta_f(\ell)=a_f(\ell)$ and $\alpha_f(\ell)\beta_f(\ell)=\ell^{2k-3}$. Set

$$D(s,\pi_f)=\prod_{\ell}\left((1-\alpha_f(\ell)^2\ell^{-s})(1-\alpha_f(\ell)\beta_f(\ell)\ell^{-s})(1-\beta_f(\ell)^2\ell^{-s})\right)^{-1}.$$

Shimura has shown this Euler product converges if the real part of s is sufficiently large and can be extended to a meromorphic function on the entire complex plane that is holomorphic except for possible simple poles at $s=2k-2$ and $2k-3$ ([35], Theorem 1). The values of $D(2k-2,\pi_f)/U(\pi_f)$ are in $\mathcal{O}$ ([19], Page 86) where

$$U(\pi_f)=\frac{(2\pi)^{2k-1}\,\Omega_f^+\Omega_f^-}{(2k-3)!}.$$

Following Hida we define $\varepsilon\in K$ by

$$a=\frac{D(2k-2,\pi_f)}{\varepsilon\cdot U(\pi_f)} \tag{12}$$

where a is given by Equation 11.

Theorem 5.3. *([19], Theorem 2.5) Let $f\in S_{2k-2}(\Gamma_1,\mathcal{O})$ be a newform. Let $\mathfrak{p}$ be the prime of $\mathcal{O}$ over p. If f is ordinary at $\mathfrak{p}$, then ε is a unit in $\mathcal{O}$.*

Combining ([17], Theorem 5.1) and ([40], 8.2.17) we have

$$D(2k-2,\pi_f)=\frac{2^{4k-4}\,\pi^{2k-1}}{(2k-3)!}\,\langle f,f\rangle.$$

Inserting this expression for $D(2k-2,\pi_f)$ into Equation 12 and simplifying we obtain

$$a=\frac{2^{2k-3}}{\varepsilon\cdot\Omega_f^+\Omega_f^-}\langle f,f\rangle.$$

Combining Equations 10 and 11 we can write

$$\mathbb{T}_{\mathcal{O},\mathfrak{m}}/(\wp_f\oplus\bigcap_{i=2}^{r}\wp_{f_i})\cong\mathcal{O}/a\mathcal{O} \tag{13}$$

where

$$a=\frac{2^{2k-3}}{\varepsilon\cdot\Omega_f^+\Omega_f^-}\langle f,f\rangle. \tag{14}$$

Since $\mathbb{T}_{\mathcal{O},\mathfrak{m}}/\wp_f\cong\mathcal{O}$, there exists a $t\in I_f$ that maps to a under the above isomorphism. Thus we have that

$$tf_i=\begin{cases} af & \text{if } i=1\\ 0 & \text{if } 2\le i\le r.\end{cases}$$

This is the Hecke operator we seek. Using the fact that

$$\mathbb{T}_{\mathcal{O}}\cong\prod\mathbb{T}_{\mathcal{O},\mathfrak{m}}$$

where the product is over the maximal ideals of $\mathbb{T}_{\mathcal{O}}$, we can view $\mathbb{T}_{\mathcal{O},\mathfrak{m}}$ as a subring of $\mathbb{T}_{\mathcal{O}}$. Therefore we have the following theorem.

Theorem 5.4. *Let $f=f_1,f_2,\dots,f_r$ be a basis of eigenforms of $S_{2k-2}(\Gamma_1,\mathcal{O})$ with $k>2$. Suppose that the representation $\rho_{\mathfrak{m}}$ associated to f is irreducible and f is ordinary at $\mathfrak{p}$. There exists a Hecke operator $t\in\mathbb{T}_{\mathcal{O}}$ such that $tf=af$ and $tf_i=0$ for $i\ge 2$ where a is as in Equation 14 with ε a unit in $\mathcal{O}$.*

6. The congruence

In this section we combine the results of the previous sections to produce a congruence between the Saito-Kurokawa lift F_f and a cuspidal Siegel eigenform G which is not a Saito-Kurokawa lift. We fix $k > 3$ throughout this section.

6.1. **Congruent to a Siegel modular form.** Let $f \in S_{2k-2}(\Gamma_1)$ be a newform and F_f the Saito-Kurokawa lift as constructed in Section 3. Recall that $\mathcal{E}(Z,W)$ is a Siegel modular form of weight k and level N in each variable. Before we go any further we need to replace $\mathcal{E}(Z,W)$ with a form of level 1. The reason for this will be clear shortly as we will need to apply a Hecke operator that is of level 1. We do this by taking the trace. Set

$$\tilde{\mathcal{E}}(Z,W) = \sum_{\gamma\times\delta\in\Gamma_2/\Gamma_0^2(N)\times\Gamma_2/\Gamma_0^2(N)} \mathcal{E}(Z,W)|_{(\gamma\times\delta)}.$$

It is clear that $\tilde{\mathcal{E}}(Z,W)$ is now a Siegel modular form on $\Gamma_2\times\Gamma_2$. The Fourier coefficients are seen to still be in $\mathbb{Z}_p[\chi]$ by applying the q-expansion principle for Siegel modular forms ([6], Prop. 1.5).

Let $F_0 = F_f, F_1, \dots F_r$ be a basis of eigenforms for the Hecke operators $T(\ell)$ ($\ell \neq p$) of $\mathcal{M}_k(\Gamma_2)$ so that F_i is orthogonal to F_f for $1 \le i \le r$. We enlarge $\mathcal{O}$ here if necessary so that
1. $\mathcal{O}$ contains the values of χ
2. the eigenforms F_i are all defined over $\mathcal{O}$
3. the newforms f_i defined in Conjecture 5.2 are defined over $\mathcal{O}$.

Following Shimura, we write

$$\tilde{\mathcal{E}}(Z,W) = \sum_{i,j} c_{i,j} F_i(Z) F_j(W) \tag{15}$$

with $c_{i,j} \in \mathbb{C}$ ([43], Eq. 7.7).

Lemma 6.1. *Equation 15 can be written in the form*

$$\tilde{\mathcal{E}}(Z,W) = c_{0,0} F_f(Z) F_f(W) + \sum_{\substack{0\le i\le r\\ 0<j\le r}} c_{i,j} F_i(Z) F_j(W).$$

Proof. Recall Shimura's inner product formula as given in Equation 9:

$$\langle \mathcal{E}(Z,W), F_f(W)\rangle_{\Gamma_0^2(N)} = \pi^{-3}\,\mathcal{A}_{k,N} L_{\mathrm{st}}^{\Sigma}(5-k, F_f, \chi) F_f(Z)$$

and observe that

$$\langle \mathcal{E}(Z,W), F_f(W)\rangle_{\Gamma_0^2(N)} = \langle \tilde{\mathcal{E}}(Z,W), F_f(W)\rangle_{\Gamma_2}$$

by the way we defined the inner product. Note that we insert the "$\Gamma_0^2(N)$" and "Γ_2" here merely to make explicit which group the inner product is defined on. On the other hand, if we take the inner product of the right hand side of Equation 15 with $F_f(W)$ we get

$$\langle \tilde{\mathcal{E}}(Z,W), F_f(W)\rangle = \sum_{0\le i\le r} c_{i,0}\langle F_f, F_f\rangle F_i(Z).$$

Equating the two we get

$$\pi^{-3}\,\mathcal{A}_{k,N} L_{\mathrm{st}}^{\Sigma}(5-k, F_f, \chi) F_f(Z) = \sum_{0\le i\le r} c_{i,0}\langle F_f, F_f\rangle F_i(Z).$$

Since the F_i form a basis, it must be the case that $c_{i,0} = 0$ unless $i = 0$, which gives the result. $\square$

Our goal is to show that we can write $c_{0,0}$ as a product of a unit in $\mathcal{O}$ and $\frac{1}{\varpi^m}$ for some $m \geq 1$. Once we have shown we can do this, it will be straightforward to move from this to the congruence we desire.

Using Lemma 6.1 and Equation 9 we write

$$c_{0,0}\langle F_f, F_f\rangle F_f(Z) = \pi^{-3}\,\mathcal{A}_{k,N} L_{\mathrm{st}}^{\Sigma}(5-k, F_f, \chi) F_f(Z). \tag{16}$$

Equating the coefficient of $F_f(Z)$ on each side and solving for $c_{0,0}$ gives us

$$c_{0,0} = \frac{\mathcal{A}_{k,N} L_{\mathrm{st}}^{\Sigma}(5-k, F_f, \chi)}{\pi^3\,\langle F_f, F_f\rangle}. \tag{17}$$

Combining Theorems 3.9 and 3.10 with Equation 17 we have

$$c_{0,0} = \mathcal{B}_{k,N} \frac{|D|^{k-3/2}\, L(k-1, f, \chi_D) L^{\Sigma}(3-k, \chi) L^{\Sigma}(1, f, \chi) L^{\Sigma}(2, f, \chi)}{\pi^2\, |c_g(|D|)|^2\, L(k, f)\langle f, f\rangle} \tag{18}$$

with

$$\mathcal{B}_{k,N} = \frac{(-1)^k 2^{2k+2}\, 3\, v_N}{(k-1)[\Gamma_2 : \Gamma_0^2(N)]}. \tag{19}$$

The main obstacle at this point to studying the ϖ-valuation of $c_{0,0}$ is the possibility that the congruence we produce would be to a Saito-Kurokawa lift. Fortunately, we can apply the results of Section 5 to remove this possibility. We will do this by applying a Hecke operator t_S to Equation 15.

Assume that Conjecture 5.2 is satisfied. Recall that we showed this is the case if f is ordinary at $\mathfrak{p}$. We have a Hecke operator $t \in \mathbb{T}_{\mathcal{O}}$ that acts on f via the eigenvalue $u\,\dfrac{\langle f, f\rangle}{\Omega_f^+ \Omega_f^-}$ for u a unit in $\mathcal{O}$ and kills f_i for all other f_i in a basis of newforms for $S_{2k-2}(\Gamma_1, \mathcal{O})$. Using that the Saito-Kurokawa correspondence is Hecke-equivariant, we have associated to t a Hecke operator $t_S \in \mathbb{T}_{S,\mathcal{O}}$ so that

$$t_S \cdot F_{f_i} = \begin{cases} u\,\frac{\langle f,f\rangle}{\Omega_f^+\Omega_f^-}\, F_f & \text{for } f_i = f \\ 0 & \text{for } f_i \neq f. \end{cases} \tag{20}$$

Applying t_S to Equation 15 as a modular form in W we obtain

$$t_S \tilde{\mathcal{E}}(Z, W) = c'_{0,0} F_f(Z) F_f(W) + \sum_{\substack{0 \leq i \leq r \\ 0 < j \leq r}} c_{i,j} F_i(Z) t_S F_j(W) \tag{21}$$

with

$$c'_{0,0} = u\,\frac{\langle f, f\rangle}{\Omega_f^+ \Omega_f^-} \cdot c_{0,0} = \mathcal{C}_{k,N}\, \frac{|D|^{k-3/2}\, L(k-1, f, \chi_D) L^{\Sigma}(3-k, \chi) L^{\Sigma}(1, f, \chi) L^{\Sigma}(2, f, \chi)}{\pi^2\, |c_g(|D|)|^2\, L(k, f)\Omega_f^+ \Omega_f^-} \tag{22}$$

where

$$\mathcal{C}_{k,N} = u \cdot \mathcal{B}_{k,N}. \tag{23}$$

Note that we have killed any F_j that is a Saito-Kurokawa lift.

Our next step is to normalize the L-values in Equation 22 so as to obtain algebraic values. Theorem 5.1 showed that if we divide $L(m, f, \chi)$ by $\tau(\chi)(2\pi i)^m \Omega_f^{\pm}$ we get

a value in $\mathcal{O}$ where we choose Ω_f^+ if $\chi(-1) = (-1)^m$ and choose Ω_f^- if $\chi(-1) = (-1)^{m-1}$. It is easy to see that if Ω_f^+ is associated to $L(1, f, \chi)$, then Ω_f^- is associated to $L(2, f, \chi)$ and vice versa. Therefore we have

$$\frac{L(1,f,\chi)L(2,f,\chi)}{\Omega_f^+\Omega_f^-} = \tau(\chi)^2(2\pi i)^3 L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi).$$

In particular, we have

$$\frac{L^\Sigma(1,f,\chi)L^\Sigma(2,f,\chi)}{\Omega_f^+\Omega_f^-} = \frac{\tau(\chi)^2(2\pi i)^3 L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)}{L_\Sigma(1,f,\chi)L_\Sigma(2,f,\chi)}.$$

Next we turn our attention to the ratio $\dfrac{L(k-1,f,\chi_D)}{L(k,f)}$. Since $L(k, f)$ has no character, we see that we associate Ω_f^+ to $L(k, f)$ if k is even and Ω_f^- if k is odd. We need to associate the same period to $L(k - 1, f, \chi_D)$. The way to accomplish this is to choose D so that $\chi_D(-1) = -1$. Therefore we have

$$\frac{L(k-1,f,\chi_D)}{L(k,f)} = \frac{\tau(\chi_D)L_{\mathrm{alg}}(k-1,f,\chi_D)}{(2\pi i)L_{\mathrm{alg}}(k,f)}.$$

Also recall that in Section 4 we showed that $L^\Sigma(3-k, \chi) \in \mathbb{Z}_p[\chi]$ for $\gcd(p, N) = 1$.

Gathering these results together we have:

$$c'_{0,0} = \mathcal{D}_{k,N,\chi,D}\mathcal{L}(k,f,D,\chi) \tag{24}$$

where

$$\mathcal{L}(k,f,D,\chi) = \frac{L^\Sigma(3-k,\chi)L_{\mathrm{alg}}(k-1,f,\chi_D)L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)}{L_{\mathrm{alg}}(k,f)}$$

and

$$\mathcal{D} := \mathcal{D}_{k,N,\chi,D} = \frac{(-1)^{k+1}\,2^{2k+4}\,3\,|D|^k\tau(\chi_D)\tau(\chi)^2}{(k-1)\,[\Gamma_2:\Gamma_0^2(N)]|D|^{3/2}\,|c_g(|D|)|^2\,L_\Sigma(1,f,\chi)L_\Sigma(2,f,\chi)}.$$

Everything in these equations is now algebraic, so it comes down to studying the ϖ-divisibility of each of the terms. We would like to show that ϖ^m divides the denominator for some $m \geq 1$ but not the numerator. Note that as long as everything in the denominator is a ϖ-integer, we do not have to worry about anything written in the denominator contributing a "ϖ" to the numerator.

We first deal with $\mathcal{D}$. We know from Conjecture 5.2 that u is a unit of $\mathcal{O}$ so long as $\rho_{\mathfrak{m}}$ is irreducible. Under this assumption we need not worry about u. Choosing p relatively prime to D takes care of the D's that appear. We also can see that $\varpi \nmid \tau(\chi)$ and $\varpi \nmid \tau(\chi_D)$. For instance, suppose $\varpi \mid \tau(\chi)$. Then this would imply that $\varpi \mid (\tau(\chi)\overline{\tau(\chi)})^2 = (\sqrt{N})^2 = N$, a contradiction and similarly for $\tau(\chi_D)$. Next we need to deal with $\dfrac{1}{L_\Sigma(1,f,\chi)L_\Sigma(2,f,\chi)}$. Observe that we can write

$$\begin{aligned}\frac{1}{L_{(\ell)}(1,f,\chi)} &= \frac{1}{(1-\lambda_f(\ell)\ell^{-1}+\ell^{2k-5})} \\ &= \frac{\ell}{(\ell-\lambda_f(\ell)+\ell^{2k-4})}.\end{aligned}$$

Since $\ell \mid N$ and $\gcd(p, N) = 1$, we have that $p \nmid \ell$. It is also clear now that the denominator is in $\mathcal{O}$. Since we can do this for each $\ell \mid N$ and the same

argument follows for $L_{(\ell)}(2, f, \chi)$, we see that $\dfrac{1}{L_\Sigma(1, f, \chi)L_\Sigma(2, f, \chi)}$ cannot possibly contribute any ϖ's to the numerator. Recall that $|c_{g_f}(|D|)|^2 \in \mathcal{O}$. Therefore, so long as we choose our $p > 2k-2$ and p relatively prime to $[\Gamma_2 : \Gamma_0^2(N)]$, we have that $\mathcal{D}$ cannot contribute any ϖ's to the numerator.

The term $\mathcal{L}(k, f, D, \chi)$ is where the divisibility assumption enters into our calculations. We assume here that for some integer $m \geq 1$ we have $\varpi^m \mid L_{\mathrm{alg}}(k, f)$ and that if $\varpi^n \parallel L^\Sigma(3-k, \chi)L_{\mathrm{alg}}(k-1, f, \chi_D)L_{\mathrm{alg}}(1, f, \chi)L_{\mathrm{alg}}(2, f, \chi)$ then $n < m$ so that we end up with a ϖ in the denominator of $c'_{0,0}$.

Under these assumptions we can write

$$t_S\tilde{\mathcal{E}}(Z, W) = \frac{A}{\varpi^{m-n}} F_f(Z)F_f(W) + \sum_{\substack{0 \leq i \leq r \\ 0 < j \leq r}} c_{i,j} F_i(Z) t_S F_j(W) \tag{25}$$

for some ϖ-unit A. Recall that Corollary 3.8 gave that F_f has Fourier coefficients in $\mathcal{O}$ and that we can find a T_0 so that $\varpi \nmid A_{F_f}(T_0)$. This allows us to immediately conclude that we must have some $c_{i,j} \neq 0$ for at least one of $i, j \neq 0$. Otherwise we would have $t_S\tilde{\mathcal{E}}(Z, W) = \frac{A}{\varpi^{m-n}} F_f(Z)F_f(W)$ and using the integrality of the Fourier coefficients of $t_S\tilde{\mathcal{E}}(Z, W)$ we would get $F_f(Z)F_f(W) \equiv 0 (\bmod \varpi^{m-n})$, a contradiction.

Recall that by Corollary 3.8 there exists a T_0 so that $A_{F_f}(T_0)$ is in $\mathcal{O}^\times$. Expand each side of Equation 25 in terms of Z, reduce modulo ϖ and equate the T_0^{th} Fourier coefficients. Using the $\mathcal{O}$-integrality of the Fourier coefficients of $t_S\tilde{\mathcal{E}}(Z, W)$ we obtain:

$$A_{F_f}(T_0)F_f(W) \equiv -\frac{\varpi^{m-n}}{A} \sum_{\substack{0 \leq i \leq r \\ 0 < j \leq r}} c_{i,j} A_{F_i}(T_0) t_S F_j(W) (\bmod \varpi^{m-n}),$$

i.e., we have a congruence $F_f \equiv G (\bmod \varpi^{m-n})$ for $G \in \mathcal{M}_k(\Gamma_1)$ where

$$G(W) = -\frac{\varpi^{m-n}}{A \cdot A_{F_f}(T_0)} \sum_{\substack{0 \leq i \leq r \\ 0 < j \leq r}} c_{i,j} A_{F_i}(T_0) t_S F_j(W). \tag{26}$$

Since the Hecke operator t_S killed all F_j $(0 < j \leq r)$ that came from Saito-Kurokawa lifts, we have that G is a sum of forms that are not Saito-Kurokawa lifts.

Momentarily we will show how G can be used to produce a non-Saito-Kurokawa cuspidal eigenform with eigenvalues that are congruent to the eigenvalues of F_f, but before we do we gather our results into the following theorem.

Theorem 6.2. *Let $k > 3$ be an integer and p a prime so that $p > 2k-2$. Let $f \in S_{2k-2}(\Gamma_1, \mathcal{O})$ be a newform with real Fourier coefficients and F_f the Saito-Kurokawa lift of f. Suppose that $\rho_{\mathfrak{m}}$ is irreducible and that Conjecture 5.2 is satisfied. If there exists an integer $N > 1$, a fundamental discriminant D so that $(-1)^{k-1}D > 0$, $\chi_D(-1) = -1$, $p \nmid ND[\Gamma_2 : \Gamma_0^2(N)]$, and a Dirichlet character χ of conductor N so that*

$$\varpi^m \mid L_{\mathrm{alg}}(k, f)$$

with $m \geq 1$ and

$$\varpi^n \parallel L^{\Sigma}(3-k,\chi)L_{\mathrm{alg}}(k-1,f,\chi_D)L_{\mathrm{alg}}(1,f,\chi)L_{\mathrm{alg}}(2,f,\chi)$$

with $n < m$, then there exists $G \in \mathcal{M}_k(\Gamma_2)$ that is a sum of eigenforms that are not Saito-Kurokawa lifts so that

$$F_f \equiv G (\operatorname{mod} \varpi^{m-n}).$$

6.2. **Congruent to a non-Saito-Kurokawa cuspidal eigenform.** In this section we will show how given a congruence

$$F_f \equiv G (\operatorname{mod} \varpi^{m}) \tag{27}$$

for $m \geq 1$ as in Theorem 6.2, we can find a non-Maass cuspidal eigenform that has the same eigenvalues as F_f modulo ϖ.

Notation 6.3. If F_1 and F_2 have eigenvalues that are congruent modulo ϖ, we will write

$$F_1 \equiv_{\mathrm{ev}} F_2 (\operatorname{mod} \varpi)$$

where the ev stands for the congruence being a congruence of eigenvalues.

We begin by showing that given a congruence as in Theorem 6.2, there must be a non-Saito-Kurokawa eigenform F so that $F \equiv_{\mathrm{ev}} F_f (\operatorname{mod} \varpi)$. Once we have shown this, we will show that we can obtain an eigenvalue congruence to a cusp form. Applying the first result again we obtain our final goal of an eigenvalue congruence between F_f and a cuspidal eigenform that is not a Saito-Kurokawa lift.

Lemma 6.4. *Let $G \in \mathcal{M}_k(\Gamma_2)$ be as in Equation 26 so that we have the congruence $G \equiv F_f (\operatorname{mod} \varpi)$. Then there exists an eigenform F so that F is not a Saito-Kurokawa lift and $F_f \equiv_{\mathrm{ev}} F (\operatorname{mod} \varpi)$.*

Proof. As in Equation 26 write $G = \sum c_i F_i$ with each F_i an eigenform and $c_i \in \mathcal{O}$. It is clear from the construction of G that $F_i \neq F_f$ and F_i is not a Saito-Kurokawa lift for all i. Recall that we have the decomposition

$$\mathbb{T}_{S,\mathcal{O}} \cong \prod \mathbb{T}_{S,\mathcal{O},\mathfrak{m}}$$

where the $\mathfrak{m}$ are maximal ideals of $\mathbb{T}_{S,\mathcal{O}}$ containing ϖ. Let $\mathfrak{m}_{F_f}$ be the maximal ideal corresponding to F_f. There is a Hecke operator $t \in \mathbb{T}_{S,\mathcal{O}}$ so that $tF_f = F_f$ and $tF = 0$ for any eigenform F that does not correspond to $\mathfrak{m}_{F_f}$, i.e., if $F \not\equiv_{\mathrm{ev}} F_f (\operatorname{mod} \varpi)$. If $F_i \not\equiv_{\mathrm{ev}} F_f (\operatorname{mod} \varpi)$ for every i then applying t to the congruence $G \equiv F_f (\operatorname{mod} \varpi)$ would then yield $F_f \equiv 0 (\operatorname{mod} \varpi)$, a contradiction to the fact that $A_{F_f}(T_0) \in \mathcal{O}^{\times}$. Thus there must be an i so that $F_f \equiv_{\mathrm{ev}} F_i (\operatorname{mod} \varpi)$. □

We now show that we actually have an eigenvalue congruence to a cusp form. Before we prove this fact, we briefly recall the Siegel operator Φ. The Siegel operator is defined by

$$\Phi(F(\tau)) = \lim_{\lambda \to \infty} F\left(\begin{pmatrix} \tau & 0 \\ 0 & i\lambda \end{pmatrix}\right)$$

where $\tau \in \mathfrak{h}^1$. In terms of Fourier coefficients we have

$$\Phi(F(\tau)) = \sum_{n \geq 0} a_F\left(\begin{pmatrix} n & 0 \\ 0 & 0 \end{pmatrix}\right) e^{2\pi i n \tau}.$$

From this expression it is clear that if F has Fourier coefficients in $\mathcal{O}$, so does $\Phi(F)$. We note the following facts about the Siegel operator which can all be found in [14]:

(1) Given a Siegel modular form $F \in \mathcal{M}_k(\Gamma_2)$, one has $\Phi(F) \in M_k(\Gamma_1)$.

(2) If $\Phi(F) = 0$, then F is a cusp form.

(3) If F is an eigenform of the operator $T_S(\ell)$, then $\Phi(F)$ is an eigenform of $T(\ell)$.

(4) We have the following formula:
$$\Phi(T_S(\ell)F) = (1 - \ell^{2-k})T(\ell)\Phi(F).$$

Let $F_f \equiv_{\text{ev}} F(\operatorname{mod} \varpi)$ with F the non Saito-Kurokawa eigenform constructed in Lemma 6.4. Suppose that F is not a cusp form so that $\Phi(F) \neq 0$. Let $g = \Phi(F)$. We denote the n^{th} eigenvalue of g as $\lambda_g(n)$. Let ℓ be a prime so that $\ell \neq p$. Note that since F has eigenvalues in $\mathcal{O}$ and (4) gives that $\lambda_F(\ell) = (1 - \ell^{2-k})\lambda_g(\ell)$, we must have $(1 - \ell^{2-k})\lambda_g(\ell) \in \mathcal{O}$. Applying (4) again gives
$$\Phi(T_S(\ell)F) = (1 - \ell^{2-k})\lambda_g(\ell)g.$$
On the other hand, the (4) and the congruence give us that
$$\begin{array}{rcl} \Phi(T_S(\ell)F) & \equiv_{\text{ev}} & \Phi(T_S(\ell)F_f)(\operatorname{mod} \varpi) \\ & = & \Phi(\lambda_{F_f}(\ell)F_f) \\ & = & \lambda_{F_f}(\ell)g \\ & = & \left(\ell^{k-1} + \ell^{k-2} + \lambda_f(\ell)\right) g. \end{array}$$
Thus we have that
$$(\ell^{k-1} + \ell^{k-2} + \lambda_f(\ell)) \equiv (1 - \ell^{2-k})\lambda_g(\ell)(\operatorname{mod} \varpi). \tag{28}$$

Denote the Galois representation associated to f by ρ_f and similarly for g. Denote the residual representations after reducing modulo ϖ by $\overline{\rho}_f$ and $\overline{\rho}_g$. Equation 28 and the Tchebotarov Density Theorem show that we have the following equivalence of 4-dimensional Galois representations
$$\begin{pmatrix} \omega^{k-1} & & \\ & \omega^{k-2} & \\ & & \overline{\rho}_f \end{pmatrix} = \begin{pmatrix} \overline{\rho}_g & \\ & \omega^{2-k}\overline{\rho}_g \end{pmatrix}.$$
It is clear from this that $\overline{\rho}_f$ must be reducible. However, we assumed before that this was not the case. This contradiction shows that $\Phi(F) = 0$. We have proved the following theorem.

Theorem 6.5. *Let $k > 3$ be an integer and p a prime so that $p > 2k - 2$. Let $f \in S_{2k-2}(\Gamma_1, \mathcal{O})$ be a newform with real Fourier coefficients and F_f the Saito-Kurokawa lift of f. Suppose that $\rho_{\mathfrak{m}}$ is irreducible and that Conjecture 5.2 is satisfied. If there exists an integer $N > 1$, a fundamental discriminant D so that $(-1)^{k-1}D > 0$, $\chi_D(-1) = -1$, $p \nmid ND[\Gamma_2 : \Gamma_0^2(N)]$, and a Dirichlet character χ of conductor N so that*
$$\varpi^m \mid L_{\text{alg}}(k, f)$$
with $m \geq 1$ and
$$\varpi^n \parallel L^{\Sigma}(3 - k, \chi)L_{\text{alg}}(k - 1, f, \chi_D)L_{\text{alg}}(1, f, \chi)L_{\text{alg}}(2, f, \chi)$$

with $n < m$, *then there exists an eigenform* $G \in \mathcal{S}_k(\Gamma_2)$ *that is not a Saito-Kurokawa lift so that*

$$F_f \equiv_{\mathrm{ev}} G (\mathrm{mod}\, \varpi).$$

7. Generalities on Selmer groups

In this section we define the relevant Selmer group following Bloch and Kato [3] and Diamond, Flach, and Guo [7]. We also collect various results that are not easily located in existing references. We conclude the section by stating a version of the Bloch-Kato conjecture for modular forms.

For a field K and a topological $\mathrm{Gal}(\overline{K}/K)$-module M, we write $\mathrm{H}^1(K, M)$ for $\mathrm{H}^1_{\mathrm{cont}}(\mathrm{Gal}(\overline{K}/K), M)$ to ease notation, where "cont" indicates continuous cocycles. We write D_ℓ to denote a decomposition group at ℓ and I_ℓ to denote an inertia group at ℓ. We identify D_ℓ with $\mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$.

Let E be a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ the ring of integers of E, and ϖ a uniformizer. Let V be a p-adic Galois representation defined over E. Let $T \subseteq V$ be a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathcal{O}$-lattice. Set $W = V/T$. For $n \geq 1$, put

$$W_n = W[\varpi^n] = \{x \in W : \varpi^n x = 0\} \cong T/\varpi^n T.$$

In the following section we will construct non-zero cohomology classes in $\mathrm{H}^1(\mathbb{Q}, W_1)$ and we would like to know that they remain non-zero when we map them into $\mathrm{H}^1(\mathbb{Q}, W)$ under the natural map.

Lemma 7.1. *If* $T/\varpi T$ *is irreducible as an* $(\mathcal{O}/\varpi\mathcal{O})\,[\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$*-module then* $\mathrm{H}^1(\mathbb{Q}, W_1)$ *injects into* $\mathrm{H}^1(\mathbb{Q}, W)$.

Proof. Consider the exact sequence

$$0 \longrightarrow W_1 \longrightarrow W \xrightarrow{\cdot\varpi} W \longrightarrow 0.$$

This short exact sequence gives rise to the long exact sequence of cohomology groups

$$0 \longrightarrow \mathrm{H}^0(\mathbb{Q}, W_1) \longrightarrow \mathrm{H}^0(\mathbb{Q}, W) \xrightarrow{\cdot\varpi} \mathrm{H}^0(\mathbb{Q}, W) \longrightarrow$$

$$\mathrm{H}^1(\mathbb{Q}, W_1) \xrightarrow{\psi} \mathrm{H}^1(\mathbb{Q}, W) \longrightarrow \cdots.$$

We show that ψ is injective. Recalling that $\mathrm{H}^0(G, M) = M^G$, it is clear that $\mathrm{H}^0(\mathbb{Q}, W_1) = 0$ since we have assumed that $T/\varpi T$ is irreducible. Since W is torsion, $\mathrm{H}^0(\mathbb{Q}, W)$ is necessarily torsion as well. If $\mathrm{H}^0(\mathbb{Q}, W)$ contains a non-zero element, multiplying by a suitable ϖ^m makes it a non-zero element in W_1. This would give us a non-zero element in $\mathrm{H}^0(\mathbb{Q}, W_1)$, a contradiction. Thus, we obtain that $\mathrm{H}^0(\mathbb{Q}, W) = 0$ and so ψ is an injection. □

We also have that $\mathrm{H}^1(\mathbb{Q}_\ell, W_1)$ injects into $\mathrm{H}^1(\mathbb{Q}_\ell, W)$ when $T/\varpi T$ is irreducible as an $(\mathcal{O}/\varpi\mathcal{O})\,[D_\ell]$-module by an analogous argument.

We write $\mathbb{B}_{\mathrm{crys}}$ to denote Fontaine's ring of p-adic periods as defined in [13]. For a p-adic representation V, set

$$D_{\mathrm{crys}} = (V \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{crys}})^{D_p}.$$

Definition 7.2. A p-adic representation V is called *crystalline* if

$$\dim_{\mathbb{Q}_p} V = \dim_{\mathbb{Q}_p} D_{\mathrm{crys}}.$$

Definition 7.3. A crystalline representation V is called *short* if the following hold
1. $\mathrm{Fil}^0 D_{\mathrm{crys}} = D_{\mathrm{crys}}$ and $\mathrm{Fil}^p D_{\mathrm{crys}} = 0$,
2. if V' is a nonzero quotient of V, then $V' \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(p-1)$ is ramified

where $\mathrm{Fil}^i D_{\mathrm{crys}}$ is a decreasing filtration of D_{crys} as given in [7].

Following Bloch-Kato ([3]), we define spaces $\mathrm{H}^1_f(\mathbb{Q}_\ell, V)$ by

$$\mathrm{H}^1_f(\mathbb{Q}_\ell, V) = \begin{cases} \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, V) & \ell \neq p, \infty \\ \ker(\mathrm{H}^1(\mathbb{Q}_p, V) \to \mathrm{H}^1(\mathbb{Q}_p, V \otimes \mathbb{B}_{\mathrm{crys}})) & \ell = p \end{cases}$$

where

$$\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, M) = \ker(\mathrm{H}^1(\mathbb{Q}_\ell, M) \to \mathrm{H}^1(I_\ell, M))$$

for any D_ℓ-module M. The Bloch-Kato groups $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$ are defined by

$$\mathrm{H}^1_f(\mathbb{Q}_\ell, W) = \mathrm{im}(\mathrm{H}^1_f(\mathbb{Q}_\ell, V) \to \mathrm{H}^1(\mathbb{Q}_\ell, W)).$$

One should note here that the f appearing in these definitions has nothing to do with the elliptic modular form f we have been working with and is merely standard notation in the literature (standing for "finite part".)

Lemma 7.4. *If V is unramified at ℓ, then*

$$\mathrm{H}^1_f(\mathbb{Q}_\ell, W) = \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W).$$

Proof. We need only show that $\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, V)$ surjects onto $\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W)$. The short exact sequence

$$0 \longrightarrow T \longrightarrow V \longrightarrow W \longrightarrow 0$$

gives rise to the long exact sequence in cohomology

$$0 \longrightarrow \mathrm{H}^0(\mathbb{F}_\ell, T) \longrightarrow \mathrm{H}^0(\mathbb{F}_\ell, V) \longrightarrow \mathrm{H}^0(\mathbb{F}_\ell, W) \longrightarrow \mathrm{H}^1(\mathbb{F}_\ell, T) \longrightarrow$$

$$\mathrm{H}^1(\mathbb{F}_\ell, V) \xrightarrow{\psi} \mathrm{H}^1(\mathbb{F}_\ell, W) \longrightarrow \mathrm{H}^2(\mathbb{F}_\ell, T) \longrightarrow \cdots$$

where we identify $\mathrm{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell)$ with D_ℓ/I_ℓ. Since $\mathrm{Gal}(\overline{\mathbb{F}}_\ell/\mathbb{F}_\ell) \cong \hat{\mathbb{Z}}$ and $\hat{\mathbb{Z}}$ has cohomological dimension 1 ([33], Chap. 5), we have that $\mathrm{H}^2(\mathbb{F}_\ell, T) = 0$, i.e., ψ is a surjection. Observing that for any D_ℓ-module M we have a natural isomorphism

$$\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, M) \cong \mathrm{H}^1(\mathbb{F}_\ell, M^{I_\ell})$$

and using the fact that V is assumed to be unramified at ℓ and so T is unramified at ℓ as well, we obtain the result. $\square$

$$0 \longrightarrow N \longrightarrow X \longrightarrow M \longrightarrow 0.$$

Remark 7.5. Let R be a ring and let M and N be R-modules. Recall that an (R-linear) extension of M by N is a short exact sequence of R-modules
There is a bijection between $\mathrm{Ext}^1_R(M,N)$ and the set of equivalence classes of extensions of M by N. Let $\alpha \in \mathrm{H}^1(\mathbb{Q}_\ell, V)$. It is known that $\mathrm{H}^1(\mathbb{Q}_\ell, V) \cong \mathrm{Ext}^1_{E[D_\ell]}(E, V)$ ([20], Theorem 6.12). Therefore we have that α corresponds to an extension X of E by V:

$$0 \longrightarrow V \longrightarrow X \longrightarrow E \longrightarrow 0.$$

For $\ell \neq p$, one has that X is an unramified representation if and only if $\alpha \in \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, V)$. If $\ell = p$, then X is a crystalline representation if and only if $\alpha \in H^1_f(\mathbb{Q}_p, V)$.

We are now in a position to define the Selmer group of interest to us.

Definition 7.6. Let W and $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$ be defined as above. The *Selmer group* of W is given by

$$\mathrm{H}^1_f(\mathbb{Q}, W) = \ker\left(\mathrm{H}^1(\mathbb{Q}, W) \to \bigoplus_\ell \frac{\mathrm{H}^1(\mathbb{Q}_\ell, W)}{\mathrm{H}^1_f(\mathbb{Q}_\ell, W)}\right),$$

i.e., it consists of the cocycles $c \in \mathrm{H}^1(\mathbb{Q}, W)$ that when restricted to D_ℓ lie in $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$ for each ℓ.

Lemma 7.4 allows us to identify $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$ with $\mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W)$ for $\ell \neq p$. Define $\mathrm{H}^1_f(\mathbb{Q}_\ell, W_n) = \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W_n)$ for $\ell \neq p$. At the prime p, we define $\mathrm{H}^1_f(\mathbb{Q}_p, W_n) \subseteq \mathrm{H}^1(\mathbb{Q}_p, W_n)$ to be the subset of classes of extensions of D_p-modules

$$0 \longrightarrow W_n \longrightarrow X \longrightarrow \mathcal{O}/\varpi^n\mathcal{O} \longrightarrow 0$$

so that X is in the essential image of $\mathbb{V}$ where $\mathbb{V}$ is the functor defined in Section 1.1 of [7]. We will not define the functor here; we will be content with stating the relevant properties that we will need. This essential image is stable under direct sums, subobjects, and quotients ([7], Section 2.1). This gives that $\mathrm{H}^1_f(\mathbb{Q}_p, W_n)$ is an $\mathcal{O}$-submodule of $\mathrm{H}^1(\mathbb{Q}_p, W_n)$. We also have that $\mathrm{H}^1_f(\mathbb{Q}_p, W_n)$ is the preimage of $\mathrm{H}^1_f(\mathbb{Q}_p, W_{n+1})$ under the natural map $\mathrm{H}^1(\mathbb{Q}_p, W_n) \to \mathrm{H}^1(\mathbb{Q}_p, W_{n+1})$. For our purposes, it will be enough to note the following fact.

Lemma 7.7. *([7], Page 670) If V is a short crystalline representation at p, T a D_p-stable lattice, and X a subquotient of $T/\varpi^n T$ that gives an extension of D_p-modules as above then the class of this extension is in $\mathrm{H}^1_f(\mathbb{Q}_p, W_n)$.*

We have a natural map $\phi_n : \mathrm{H}^1(\mathbb{Q}_p, W_n) \to \mathrm{H}^1(\mathbb{Q}_p, W)$. On the level of extensions this map is given by pushout via the map $\varpi^{-n}T/T \to V/T$, pullback via the map $\mathcal{O} \to \mathcal{O}/\varpi^n\mathcal{O}$, and the isomorphism $\mathrm{H}^1(\mathbb{Q}_p, W) \cong \mathrm{Ext}^1_{\mathcal{O}[D_p]}(\mathcal{O}, V/T)$. In the next section we will be interested in the situation where we have a non-zero cocycle $h \in \mathrm{H}^1(\mathbb{Q}, W_1)$ that restricts to be in $\mathrm{H}^1_f(\mathbb{Q}_\ell, W_1)$. We want to be able to conclude that this gives a non-zero cocycle in $\mathrm{H}^1(\mathbb{Q}, W)$ that restricts to be in

$\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$. We saw above that $\mathrm{H}^1(\mathbb{Q}, W_1)$ injects into $\mathrm{H}^1(\mathbb{Q}, W)$, so it only remains to show that the restriction is in $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$. This is accomplished via the following proposition.

Proposition 7.8. *([7], Prop. 2.2) The natural isomorphism*

$$\varinjlim_n \mathrm{H}^1(\mathbb{Q}_\ell, W_n) \cong \mathrm{H}^1(\mathbb{Q}_\ell, W)$$

induces isomorphisms

$$\varinjlim_n \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W_n) \cong \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W)$$

and

$$\varinjlim_n \mathrm{H}^1_f(\mathbb{Q}_p, W_n) \cong \mathrm{H}^1_f(\mathbb{Q}_p, W).$$

This proposition shows that the map ϕ_n gives a map from $\mathrm{H}^1_f(\mathbb{Q}_p, W_n)$ to $\mathrm{H}^1_f(\mathbb{Q}_p, W)$. We summarize with the following proposition.

Proposition 7.9. *Let h be a non-zero cocycle in $\mathrm{H}^1(\mathbb{Q}, W_1)$ and assume that $T/\varpi T$ is irreducible. If $h|_{D_\ell} \in \mathrm{H}^1_f(\mathbb{Q}_\ell, W_1)$ is non-zero, then $h|_{D_\ell}$ gives a non-zero ϖ-torsion element of $\mathrm{H}^1_f(\mathbb{Q}_\ell, W)$. If $h|_{D_\ell} \in \mathrm{H}^1_f(\mathbb{Q}_\ell, W_1)$ for every prime ℓ, then h is a non-zero ϖ-torsion element of $\mathrm{H}^1_f(\mathbb{Q}, W)$.*

We conclude this section with a brief discussion of the Bloch-Kato conjecture for modular forms. The reader interested in more details or a more general framework should consult [3] where the conjecture is referred to as the “Tamagawa number conjecture”.

For each prime p let $V_p := V_{f,\mathfrak{p}}$ be the p-adic Galois representation arising from a newform f of weight $2k-2$, $T_p := T_{f,\mathfrak{p}}$ a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice, and $W_p := W_{f,\mathfrak{p}} = V_p/T_p$. The W_p here should not be confused with our use of W_n earlier. Denote the j^{th} Tate twist of W_p by $W_p(j)$. Let π_* be the natural map $\mathrm{H}^1(\mathbb{Q}, V_p(j)) \to \mathrm{H}^1(\mathbb{Q}, W_p(j))$ used to define the groups $\mathrm{H}^1_f(\mathbb{Q}_\ell, W_p(j))$. We define the Tate-Shafarevich group to be

$$Ш(j) = \bigoplus_\ell \mathrm{H}^1_f(\mathbb{Q}, W_\ell(j))/\pi_* \mathrm{H}^1_f(\mathbb{Q}, V_\ell(j)). \tag{29}$$

Define the set $\Gamma_{\mathbb{Q}}(j)$ by

$$\Gamma_{\mathbb{Q}}(j) = \bigoplus_\ell \mathrm{H}^0(\mathbb{Q}, W_\ell(j)).$$

One should think of these as the analogue of the rational torsion points on an elliptic curve. Accordingly, the set $\Gamma_{\mathbb{Q}}(j)$ is often referred to as the “global points”.

Conjecture 7.10. (Bloch-Kato) With the notation as above, one has

$$L(k,f) = \frac{(\prod_\ell c_\ell(k))\, \mathrm{vol}_\infty(k) \# Ш(1-k)}{\#\Gamma_{\mathbb{Q}}(k)\#\Gamma_{\mathbb{Q}}(k-2)} \tag{30}$$

where $c_p(j)$ are “Tamagawa factors” and $\mathrm{vol}_\infty(k)$ is a certain real period. See [10] for a careful treatment of $\mathrm{vol}_\infty(k)$.

Remark 7.11. 1. It is known that away from the central critical value the Selmer group is finite ([21], Theorem 14.2). Therefore we can identify the ϖ-part of the Selmer group with the ϖ-part of the Tate-Shafarevich group.
2. If $T_\ell/\varpi T_\ell$ is irreducible, then $\mathrm{H}^0(\mathbb{Q}, W_\ell(j)) = 0$.
3. The Tamagawa factors are integers. See ([3]) for definitions and discussion.
4. The real period $\mathrm{vol}_\infty(k)$ is $\pi^k \Omega_f^{\pm}$ up to p-adic unit with the $\pm$ depending on the parity of k ([11]).

In the next section we will prove that if $\varpi \mid L_{\mathrm{alg}}(k, f)$, then $p \mid \# \mathrm{H}^1_f(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. Using Remark 7.11 this divisibility gives evidence for the Bloch-Kato conjecture as stated. In particular, we will have that if a prime ϖ divides the left hand side of Equation 30, then it divides the right hand side as well.

8. Galois Arguments

In this section we will combine the results of the previous sections to imply a divisibility result on the Selmer group $\mathrm{H}^1_f(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. Note that in this section entries of matrices denoted by $*$'s can be anything and are assumed to be of the appropriate size. Similarly, a 1 as a matrix entry is assumed to be of the appropriate size. A blank space in a matrix is assumed to be 0. We begin by stating two theorems that are fundamental to the results in this section.

Theorem 8.1. *([45], Theorem 3.1.3) Let $F \in \mathcal{S}_k(\Gamma_0^2(M))$ be an eigenform, K_F the number field generated by the Hecke eigenvalues of F, and $\mathfrak{p}$ a prime of K_F over p. There exists a finite extension E of the completion $K_{F,\mathfrak{p}}$ of K_F at $\mathfrak{p}$ and a continuous semi-simple Galois representation*

$$\rho_{F,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(E)$$

unramified at all primes $\ell \nmid pM$ so that for all $\ell \nmid pM$, we have

$$\det(X \cdot I - \rho_{F,\mathfrak{p}}(\mathrm{Frob}_\ell)) = L_{\mathrm{spin},(\ell)}(X)$$

(we are using arithmetic Frobenius here as opposed to geometric which is more prevalent in the literature.)

Theorem 8.2. *([12], [49]) Let F be as in Theorem 8.1. The restriction of $\rho_{F,\mathfrak{p}}$ to the decomposition group D_p is crystalline at p. In addition if $p > 2k-2$ then $\rho_{F,\mathfrak{p}}$ is short.*

Recall that for a Saito-Kurokawa lift one has a decomposition of the Spinor L-function: for F_f we have

$$L_{\mathrm{spin}}(s, F_f) = \zeta(s-k+1)\zeta(s-k+2)L(s,f).$$

This decomposition gives us that the Galois representation $\rho_{F_f,\mathfrak{p}}$ has a very simple form. In particular, using that $\rho_{F_f,\mathfrak{p}}$ is semi-simple and applying the Brauer-Nesbitt theorem we have that

$$\rho_{F_f,\mathfrak{p}} = \begin{pmatrix} \varepsilon^{k-2} & & \\ & \rho_{f,\mathfrak{p}} & \\ & & \varepsilon^{k-1} \end{pmatrix}$$

where ε is the p-adic cyclotomic character.

Under the conditions of Theorem 6.5, we have a non-Saito-Kurokawa cuspidal Siegel eigenform G such that $G \equiv_{\mathrm{ev}} F_f \pmod{\varpi}$. This gives a congruence between the Hecke polynomials of the Spinor L-functions of F_f and G as well. Let $\mathfrak{p}$ be

a prime of a sufficiently large finite extension $E/\mathbb{Q}_p$ so that $\mathcal{O}_E$ contains the $\mathcal{O}$ needed for the congruence and so that $\rho_{F_f,\mathfrak{p}}$ and $\rho_{G,\mathfrak{p}}$ are defined over $\mathcal{O}_E$. We set $\mathcal{O} = \mathcal{O}_E$ and let ϖ be a uniformizer of $\mathcal{O}$. Applying the Brauer-Nesbitt theorem we obtain that the semi-simplification of $\overline{\rho}_{G,\mathfrak{p}}$ is given by

$$\overline{\rho}_{G,\mathfrak{p}}^{\mathrm{ss}} = \overline{\rho}_{F_f,\mathfrak{p}} = \begin{pmatrix} \omega^{k-2} & & \\ & \overline{\rho}_{f,\mathfrak{p}} & \\ & & \omega^{k-1} \end{pmatrix}$$

where we use ω to denote the reduction of the cyclotomic character ε modulo ϖ. The goal now is to use this information on the semi-simplification of $\overline{\rho}_{G,\mathfrak{p}}$ to deduce the form of $\overline{\rho}_{G,\mathfrak{p}}$.

Our first step is to show that there is a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice T so that the reduction of $\rho_{G,\mathfrak{p}}$ is of the form

$$\overline{\rho}_{G,\mathfrak{p}} = \begin{pmatrix} \omega^{k-2} & *_1 & *_2 \\ *_3 & \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ & & \omega^{k-1} \end{pmatrix}$$

where either $*_1$ or $*_3$ is zero. We proceed by brute force, working our way backwards from the definition of the semi-simplification. We begin by noting some conjugation formulas that will be important. Expanding on the notation used in [32], write

$$P_1 = \begin{pmatrix} \varpi & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 1 & & & \\ & \varpi & & \\ & & \varpi & \\ & & & 1 \end{pmatrix},$$

and

$$P_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \varpi \end{pmatrix}.$$

We have the following conjugation formulas

$$P_1 \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ \varpi a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ \varpi a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ \varpi a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} P_1^{-1} = \begin{pmatrix} a_{1,1} & \varpi a_{1,2} & \varpi a_{1,3} & \varpi a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}, \tag{31}$$

$$P_2 \begin{pmatrix} a_{1,1} & \varpi a_{1,2} & \varpi a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & \varpi a_{4,2} & \varpi a_{4,3} & a_{4,4} \end{pmatrix} P_2^{-1} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ \varpi a_{2,1} & a_{2,2} & a_{2,3} & \varpi a_{2,4} \\ \varpi a_{3,1} & a_{3,2} & a_{3,3} & \varpi a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}, \tag{32}$$

and

$$P_3 \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \varpi a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & \varpi a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & \varpi a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} P_3^{-1} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ \varpi a_{4,1} & \varpi a_{4,2} & \varpi a_{4,3} & a_{4,4} \end{pmatrix}. \tag{33}$$

The definition of semi-simplification gives us vector spaces

$$V := V_{G,\mathfrak{p}} = V_0 \supset V_1 \supset V_2 \supset V_3 = 0$$

with each of V_0/V_1, V_1/V_2, and V_2 irreducible components of $\overline{\rho}_{G,\mathfrak{p}}$. Since we know $\overline{\rho}_{G,\mathfrak{p}}^{\mathrm{ss}}$ explicitly, we can say that V_0/V_1, V_1/V_2, and V_2 consist of two 1-dimensional spaces and one 2-dimensional space, corresponding to $\omega^{k-1}, \omega^{k-2}$ and $\overline{\rho}_{f,\mathfrak{p}}$. The difficulty is that we do not know which V_i/V_{i+1} corresponds to which of ω^{k-1}, ω^{k-2}, and $\overline{\rho}_{f,\mathfrak{p}}$. We handle this by considering all possible situations and seeing what this implies for the form of $\overline{\rho}_{G,\mathfrak{p}}$. We split this into several cases.

Case 1: $\dim V_2 = 1 = \dim V_0/V_1$, $\dim V_1/V_2 = 2$.

Case 2: $\dim V_2 = 2$, $\dim V_0/V_1 = \dim V_1/V_2 = 1$.

Case 3: $\dim V_2 = \dim V_1/V_2 = 1$, $\dim V_0/V_1 = 2$.

Each of these cases can be analyzed via the conjugation formulas given above. We illustrate this with Case 2. This case corresponds to the situation where we have either

$$\overline{\rho}_{G,\mathfrak{p}} = \begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & * & * \\ & \omega^{k-2} & * \\ & & \omega^{k-1} \end{pmatrix}$$

or

$$\overline{\rho}_{G,\mathfrak{p}} = \begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & * & * \\ & \omega^{k-1} & * \\ & & \omega^{k-2} \end{pmatrix}.$$

The first of these is handled by observing

$$\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & * & * \\ & \omega^{k-2} & * \\ & & \omega^{k-1} \end{pmatrix} \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \omega^{k-2} & & * \\ * & \overline{\rho}_{f,\mathfrak{p}} & * \\ & & \omega^{k-1} \end{pmatrix}.$$

The second is handled similarly:

$$\begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} \begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & * & * \\ & \omega^{k-1} & * \\ & & \omega^{k-2} \end{pmatrix} \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}^{-1} = \begin{pmatrix} \omega^{k-2} & & \\ * & \overline{\rho}_{f,\mathfrak{p}} & * \\ * & & \omega^{k-1} \end{pmatrix}.$$

Next we change bases as in Equation 32 and then as in Equation 33 to obtain

$$\overline{\rho}_{G,\mathfrak{p}} = \begin{pmatrix} \omega^{k-2} & * & * \\ & \overline{\rho}_{f,\mathfrak{p}} & * \\ & & \omega^{k-1} \end{pmatrix}.$$

Therefore we have that there is a lattice so that we have

$$\overline{\rho}_{G,\mathfrak{p}} = \begin{pmatrix} \omega^{k-2} & *_1 & *_2 \\ *_3 & \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ & & \omega^{k-1} \end{pmatrix}$$

where either $*_1$ or $*_3$ is zero.

Now that we have the matrix in the appropriate form, we would like to further limit the possibilities. We begin with the following proposition.

Proposition 8.3. *Let $\rho_{G,\mathfrak{p}}$ be such that it does not have a sub-quotient of dimension 1 and $\overline{\rho}_{G,\mathfrak{p}}^{\mathrm{ss}} = \omega^{k-2} \oplus \overline{\rho}_{f,\mathfrak{p}} \oplus \omega^{k-1}$. Then there exists a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable $\mathcal{O}$-lattice in V_G having an $\mathcal{O}$-basis such that the corresponding representation $\rho = \rho_{G,\mathfrak{p}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_4(\mathcal{O})$ has reduction of the form*

$$\overline{\rho}_{G,\mathfrak{p}} = \begin{pmatrix} \omega^{k-2} & *_1 & *_2 \\ *_3 & \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ & & \omega^{k-1} \end{pmatrix} \tag{34}$$

and such that there is no matrix of the form

$$U = \begin{pmatrix} 1 & & & n_1 \\ & 1 & & n_2 \\ & & 1 & n_3 \\ & & & 1 \end{pmatrix} \in \mathrm{GL}_4(\mathcal{O}) \tag{35}$$

such that $\rho' = U\rho U^{-1}$ has reduction of type (34) with $*_2 = *_4 = 0$.

Proof. Fix a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice and an $\mathcal{O}$-basis giving rise to a representation ρ_0 of type (34). Suppose there exists a U_0 as in (35). Inductively we define a converging sequence of matrices M_i so that $M_i \rho_0 M_i^{-1}$ is a representation into $\mathrm{GL}_4(\mathcal{O})$ with reduction of the form (34). Set $M_1 = U_0$. By assumption we have that $M_1 \rho_0 M_1^{-1}$ is of the required form. Define M_{i+1} inductively by $M_{i+1} = P_3^{-i} U_0 P_3^i M_i$. We have that

$$M_{i+1} = \begin{pmatrix} 1 & 0 & 0 & n_1 \sum_{n=1}^{i} \varpi^n \\ 0 & 1 & 0 & n_2 \sum_{n=1}^{i} \varpi^n \\ 0 & 0 & 1 & n_3 \sum_{n=1}^{i} \varpi^n \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From this it is clear that M_i converges to some $M_\infty \in \mathrm{GL}_4(\mathcal{O})$ of the form

$$M_\infty = \begin{pmatrix} 1 & 0 & 0 & t_1 \\ 0 & 1 & 0 & t_2 \\ 0 & 0 & 1 & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $t_j = n_j \lim_{i \to \infty} \sum_{n=1}^{i} \varpi^n$. Suppose we have that $M_i \rho_0 M_i^{-1}$ is of the required form. Using the defintion of M_{i+1} we have that $M_{i+1} \rho_0 M_{i+1}^{-1}$ is of the form that the first three entries of the rightmost column are all divisible by ϖ^i since $P_3^i M_{i+1} \rho_0 M_{i+1}^{-1} P_3^{-i}$ has entries in $\mathcal{O}$. Thus, $\rho_\infty = M_\infty \rho_0 M_\infty^{-1}$ is such that the first three entries of the rightmost column are zero. This gives a 1-dimensional subquotient of $\rho_{G,\mathfrak{p}}$, a contradiction. Thus no such U_0 can exist. $\square$

In light of this proposition our next step is to show that $\rho_{G,\mathfrak{p}}$ does not have a sub-quotient of dimension 1 as in Theorem 6.5. There are three possibilities for how $\rho_{G,\mathfrak{p}}$ could split up with a sub-quotient of dimension 1. It could have a sub-quotient of dimension 3 and of dimension 1, a 2-dimensional sub-quotient and two 1-dimensional ones, or four 1-dimensional sub-quotients. The case of a 3 dimensional sub-quotient cannot occur, see ([48], Page 512) or ([45], Proof of Theorem 3.2.1). The case of splitting into four 1-dimensional sub-quotients is not possible either. Indeed, if $\rho_{G,\mathfrak{p}} = \chi_1 \oplus \chi_2 \oplus \chi_3 \oplus \chi_4$ for characters χ_i, then $\overline{\rho}_{G,\mathfrak{p}}$ splits into four 1-dimensional sub-quotients as well but this gives a contradiction as we know $\overline{\rho}_{f,\mathfrak{p}}$ is not completely reducible ([32], Prop. 2.1).

The last case to worry about is if $\rho_{G,\mathfrak{p}}$ splits into a 2-dimensional sub-quotient and two 1-dimensional sub-quotients. In this case G must be a CAP form ([45], Proof of Theorem 3.2.1) induced from the Siegel parabolic. However, the results of [30] imply that G must then be a Saito-Kurokawa lift, a contradiction.

Summarizing to this point, we now have that there exists a $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-stable lattice $T_{G,\mathfrak{p}}$ so that the reduction $\overline{\rho}_{G,\mathfrak{p}}$ is of the form

$$\overline{\rho}_{G,\mathfrak{p}} = \begin{pmatrix} \omega^{k-2} & *_1 & *_2 \\ *_3 & \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ & & \omega^{k-1} \end{pmatrix}$$

where $*_1$ or $*_3$ is zero and so that $\overline{\rho}_{G,\mathfrak{p}}$ is not equivalent to a representation with $*_2$ and $*_4$ both zero. Write $W_{G,\mathfrak{p}}$ for $V_{G,\mathfrak{p}}/T_{G,\mathfrak{p}}$.

We now show that $*_4$ gives us a non-zero class in $\mathrm{H}^1_f(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. Note that the fact that $\overline{\rho}_{G,\mathfrak{p}}$ is a homomorphism gives that $*_4$ necessarily gives a cohomology class in $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)[\varpi])$.

First we suppose we are in the situation where $*_3 = 0$. Our first step is to show that the quotient extension

$$\begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ 0 & \omega^{k-1} \end{pmatrix} \tag{36}$$

is not split. Suppose it is split. Then by Proposition 8.3 we know that the extension

$$\begin{pmatrix} \omega^{k-2} & *_2 \\ 0 & \omega^{k-1} \end{pmatrix} \tag{37}$$

cannot be split as well. We show this gives a contradiction by showing it gives a non-trivial quotient of the ω^{-1}-isotypical piece of the p-part of the class group of $\mathbb{Q}(\mu_p)$. However, Herbrand's Theorem (see for example, [51], Theorem 6.17) says that we must then have $p \mid B_2 = \frac{1}{30}$, which clearly cannot happen.

Consider the non-split representation

$$\overline{\rho} = \begin{pmatrix} \omega^{-1} & h \\ 0 & 1 \end{pmatrix}$$

which arises from twisting the non-split representation

$$\begin{pmatrix} \omega^{k-2} & *_2 \\ 0 & \omega^{k-1} \end{pmatrix}$$

by ω^{1-k}. Note that $\overline{\rho}$ is unramified away from p because $\overline{\rho}_{G,\mathfrak{p}}$ is unramified away from p.

We claim that this representation gives us a non-trivial finite unramified abelian p-extension $K/\mathbb{Q}(\mu_p)$ with the action of $\mathrm{Gal}(K/\mathbb{Q})$ on $\mathrm{Gal}(K/\mathbb{Q}(\mu_p))$ given by ω^{-1}.

Note that $\mathbb{Q}(\mu_p) = \overline{\mathbb{Q}}^{\ker \omega^{-1}}$, so when we restrict $\overline{\rho}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p))$ we get

$$\overline{\rho}\mid_{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p))} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix},$$

i.e., we get a non-trivial homomorphism $h : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p)) \to \mathbb{F}$ where $\mathbb{F}$ is a finite field of characteristic p. Set $K = \mathbb{Q}(h) = \overline{\mathbb{Q}}^{\ker h}$, the splitting field of h.

The fact that $\mathrm{Gal}(K/\mathbb{Q}(\mu_p))$ is abelian of p-power order follows from the fact that

$$\begin{aligned} \mathrm{Gal}(K/\mathbb{Q}(\mu_p)) &\cong \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p))/\,\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(h)) \\ &= \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\mu_p))/\ker h \\ &\cong \mathrm{Image}(h) \end{aligned}$$

and $\mathrm{Image}(h)$ is a subgroup of $\mathbb{F}$, which is of p-power order. The fact that $K/\mathbb{Q}(\mu_p)$ is unramified away from p also follows easily from the fact that $\overline{\rho}$ is unramified away from p. This shows that $h(I_\ell) = 0$ for all $\ell \neq p$. In particular, $h(I_\ell(K/\mathbb{Q}(\mu_p))) = 0$ for all $\ell \neq p$. Since we have the isomorphism above to a subgroup of $\mathbb{F}$, it must be that $I_\ell(K/\mathbb{Q}(\mu_p)) = 1$ for all $\ell \neq p$.

The fact that $\mathrm{Gal}(K/\mathbb{Q})$ acts on $\mathrm{Gal}(K/\mathbb{Q}(\mu_p))$ via ω^{-1} follows from the fact that for $\sigma \in \mathrm{Gal}(K/\mathbb{Q}(\mu_p))$ and $g \in \mathrm{Gal}(K/\mathbb{Q})$, we have

$$\overline{\rho}(g\sigma g^{-1}) = \overline{\rho}(g)\overline{\rho}(\sigma)\overline{\rho}(g^{-1}),$$

i.e., we have

$$h(g\sigma g^{-1}) = \omega^{-1}(g)h(\sigma).$$

Our next step is to show that the extension $K/\mathbb{Q}(\mu_p)$ that we have constructed is actually unramified at p. We have that $h|_{D_p} \in \mathrm{H}^1(\mathbb{Q}_p, \mathbb{F}(-1))$. Therefore, we have that h gives an extension X of $\mathcal{O}/\varpi\mathcal{O}$ by $\mathbb{F}(-1)$:

$$0 \longrightarrow \mathbb{F}(-1) \longrightarrow X \longrightarrow \mathcal{O}/\varpi\mathcal{O} \longrightarrow 0.$$

Applying Lemma 8.2 and Lemma 7.7 we have that $h|_{D_p} \in \mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{F}(-1))$. A calculation in [3] shows that $\mathrm{H}^1_f(\mathbb{Q}_p, E(-1)) = 0$ where E is the field of definition for $\rho_{G,\mathfrak{p}}$. Actually, it is shown that $\mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{Q}_p(r)) = 0$ for every $r < 0$; this implies $\mathrm{H}^1_f(\mathbb{Q}_p, E(-1)) = 0$ since E is a finite extension ([3], Example 3.9). Since we define $\mathrm{H}^1_f(\mathbb{Q}_p, E/\mathcal{O}(-1))$ to be the image of the $\mathrm{H}^1_f(\mathbb{Q}_p, E(-1))$, we have $\mathrm{H}^1_f(\mathbb{Q}_p, E/\mathcal{O}(-1)) = 0$. Since $h|_{D_p} \in \mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{F}(-1))$, Proposition 7.9 gives that $h|_{D_p} \in \mathrm{H}^1_f(\mathbb{Q}_p, E/\mathcal{O}(-1))$ and hence is 0. Thus we have that h vanishes on the entire decomposition group D_p; in particular, it must be unramified at p as claimed.

Therefore, we have an unramified extension K of $\mathbb{Q}(\mu_p)$ that is of p-power order such that $\mathrm{Gal}(K/\mathbb{Q})$ acts via ω^{-1}. Let C be the p-part of the class group of $\mathbb{Q}(\mu_p)$. Class field theory tells us that we have

$$C/C^p \cong \mathrm{Gal}(F/\mathbb{Q}(\mu_p))$$

where F is the maximal unramified elementary abelian p-extension of $\mathbb{Q}(\mu_p)$. Therefore we have that $\mathrm{Gal}(K/\mathbb{Q}(\mu_p))$ is a non-trivial subgroup of the ω^{-1}-isotypical piece of the p-part of the class group of $\mathbb{Q}(\mu_p)$, a contradiction as observed above.

Therefore, we must have that the quotient extension

$$\begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ 0 & \omega^{k-1} \end{pmatrix}$$

is not split if $*_3 = 0$.

Now suppose that $*_1 = 0$. Then the extension

$$\begin{pmatrix} \omega^{k-2} & *_2 \\ 0 & \omega^{k-1} \end{pmatrix}$$

is a quotient extension and as above must necessarily be split. Therefore again we get that the subextension

$$\begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ 0 & \omega^{k-1} \end{pmatrix}$$

cannot be split.

It remains to show that $*_4$ actually lies in $\mathrm{H}^1_f(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$ since we have shown it is not zero. Write $h = *_4$ to ease notation. As noted above, we have that h gives a non-zero class in $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)[\varpi])$. Recall that in the previous section we showed that $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)[\varpi])$ injects in $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. Therefore, we have that h gives a non-zero class in $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. It remains to show that $h|_{D_\ell} \in \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W_{f,\mathfrak{p}}(1-k))$ for each $\ell \neq p$ and $h|_{D_p} \in \mathrm{H}^1_f(\mathbb{Q}_p, W_{f,\mathfrak{p}}(1-k))$. The fact that $h|_{D_\ell} \in \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W_{f,\mathfrak{p}}(1-k)[\varpi])$ for $\ell \neq p$ is clear from the fact that $\rho_{G,\mathfrak{p}}$ is unramified away from p. Therefore, we can appeal to Proposition 7.8 to obtain that $h \in \mathrm{H}^1_{\mathrm{ur}}(\mathbb{Q}_\ell, W_{f,\mathfrak{p}}(1-k))$ for all $\ell \neq p$.

The case at p is easily handled by appealing to our work in the previous section. Since $h|_{D_p} \in \mathrm{H}^1(\mathbb{Q}_p, W_{f,\mathfrak{p}}(1-k)[\varpi])$, we get an extension X of $\mathcal{O}/\varpi\mathcal{O}$ by $W_{f,\mathfrak{p}}(1-k)[\varpi]$:

$$0 \longrightarrow W_{f,\mathfrak{p}}(1-k)[\varpi] \longrightarrow X \longrightarrow \mathcal{O}/\varpi\mathcal{O} \longrightarrow 0.$$

Appealing to Lemma 8.2 and Lemma 7.7 we have that $h|_{D_p}$ lies in $\mathrm{H}^1_f(\mathbb{Q}_p, W_{f,\mathfrak{p}}(1-k)[\varpi])$ as desired. Proposition 7.9 gives that $h|_{D_p}$ lies in $\mathrm{H}^1_f(\mathbb{Q}_p, W_{f,\mathfrak{p}}(1-k))$.

Therefore, we have that h is a non-zero torsion element of $\mathrm{H}^1(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$ that lies in $\mathrm{H}^1_f(\mathbb{Q}_\ell, W_{f,\mathfrak{p}}(1-k))$ for every ℓ. Applying Proposition 7.9 to h we have that h is a non-zero ϖ-torsion element of $\mathrm{H}^1_f(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. Therefore, it must be that $p \mid \# \mathrm{H}^1_f(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k))$. We summarize with the following theorem.

Theorem 8.4. *Let $k > 3$ be an integer and $p > 2k-2$ a prime. Let $f \in S_{2k-2}(\mathrm{SL}_2(\mathbb{Z}), \mathcal{O})$ be a newform with real Fourier coefficients so that $\rho_{\mathfrak{m}_f}$ is irreducible and Conjecture 5.2 holds (for instance, if f is ordinary at p). Let*

$$\varpi^m \mid L_{\mathrm{alg}}(k, f)$$

with $m \geq 1$. If there exists an integer $N > 1$, a fundamental discriminant D, and a Dirichlet character χ of conductor N so that $(-1)^{k-1}D > 0$, $\chi_D(-1) = -1$, $p \nmid ND[\Gamma_2 : \Gamma_0^2(N)]$, and

$$\varpi^n \parallel L^{\Sigma}(3-k, \chi) L_{\mathrm{alg}}(k-1, f, \chi_D) L_{\mathrm{alg}}(1, f, \chi) L_{\mathrm{alg}}(2, f, \chi)$$

with $n < m$, then

$$p \mid \# \mathrm{H}^1_f(\mathbb{Q}, W_{f,\mathfrak{p}}(1-k)).$$

9. Numerical Example

In this concluding section we provide a numerical example of Theorem 8.4. We used the computer software MAGMA, Stein's Modular Forms Database ([46]), and Dokchitser's PARI program ComputeL ([8]).

Let $p = 516223$. We consider level 1 and weight 54 newforms in $S_{54}(\mathrm{SL}_2(\mathbb{Z}))$. There is one Galois conjugacy class of such newforms, consisting of four newforms which we label f_1, f_2, f_3, f_4. Using the software Stein wrote for MAGMA we find that

$$p \mid \prod_{i=1}^{4} L_{\mathrm{alg}}(28, f_i). \tag{38}$$

The q-expansions of each f_i are defined over a number field K_i. Appealing to MAGMA again we find each K_i is generated by a root of

$$\begin{aligned} g(x) = x^4 &+ 68476320x^3 - 19584715019010048x^2 \\ &- 1083312724663448929712l280x \\ &+ 39446133467662904714689328971776. \end{aligned}$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $g(x)$. Note that two of the α_i are real and the other two are a complex conjugate pair. Relabelling the f_i if necessary, we may assume $K_i = \mathbb{Q}(\alpha_i)$. Let $\mathcal{O}_{K_i}$ be the ring of integers of K_i. Note that $L_{\mathrm{alg}}(28, f_i) \in \mathcal{O}_{K_i}$ for each i. Therefore, using Equation 38 we see that there exists $j \in \{1,2,3,4\}$ and a prime $\wp_j \subset \mathcal{O}_{K_j}$ over p so that $\wp_j \mid L_{\mathrm{alg}}(28, f_j)$. Since the f_i are all Galois conjugate, there is a conjugate prime $\wp_i \subset \mathcal{O}_{K_i}$ over p for each $i \in \{1,2,3,4\}$ so that $\wp_i \mid L_{\mathrm{alg}}(28, f_i)$.

Let $\chi = \chi_{-3}$ where we define χ_{-3} as in [34]. It is easy to check that this χ and $D = -3$ satisfy the conditions of Theorem 8.4. Using MAGMA we find that

$$p \nmid \prod_{i=1}^{4} L_{\mathrm{alg}}(j, f_i, \chi),$$

for $j = 1, 2$ and

$$p \nmid \prod_{i=1}^{4} L_{\mathrm{alg}}(27, f_i, \chi_D).$$

We use ComputeL to show that

$$p \nmid L^{(3)}(-25, \chi).$$

In particular, this shows we satisfy the divisibility hypotheses of Theorem 8.4 for $m = 1$ and $n = 0$.

Let $F_i = K_{i,\wp_i}$ with ring of integers $\mathcal{O}_i$ and uniformizer ϖ_i. Set $\mathbb{F}_i = \mathcal{O}_i/\varpi_i = \mathbb{F}_p[\overline{\alpha}_i]$ where $\overline{\alpha}_i = \alpha_i (\mathrm{mod}\, \wp_i)$. Let $\rho_i : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathcal{O}_i)$ be the Galois representation associated to f_i. This representation is unramified away from p and crystalline at p. Let $\overline{\rho}_i : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{F}_i)$ be the residual representation. Suppose that $\overline{\rho}_i$ is reducible. Standard arguments show that $\overline{\rho}_i$ is non-split and we can write

$$\overline{\rho}_i = \begin{pmatrix} \varphi & * \\ 0 & \psi \end{pmatrix}$$

with $* \neq 0$ (see [32]). Let $\omega : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{F}_p^\times$ be the mod p cyclotomic character. Since $\varphi\psi = \omega^{53}$ and φ and ψ are necessarily unramified away from p and of order prime to p, we can write $\varphi = \omega^a$ and $\psi = \omega^b$ with $0 \leq a < b < p-1$, and $a+b = 53$ or $a+b = p-1+53$. Arguing as in the previous section where we proved that

$$\begin{pmatrix} \overline{\rho}_{f,\mathfrak{p}} & *_4 \\ 0 & \omega^{k-1} \end{pmatrix}$$

cannot be split, we have that $*$ gives a non-zero cocycle class in $\mathrm{H}^1(\mathbb{Q}, \mathbb{F}_i(a-b))$ since $a - b < 0$. As before, this shows that we must have that p divides the class number of $\mathbb{Q}(\mu_p)$, i.e., $p \mid B_{b-a+1}$ where we recall that B_n is the n^{th} Bernoulli number ([51], Theorem 6.17). Appealing to the tables of Buhler ([5]), we see that the only Bernoulli number that 516223 divides is B_{451304}. Therefore, we must have $b - a + 1 = 451304$, which in turn implies that $a + b = p - 1 + 53$ since necessarily $a > 0$. Solving this system of equations for a and b we get $a = 32486$ and $b = 483789$. Observe that we have

$$\begin{aligned} \mathrm{Tr}(\overline{\rho}_i(\mathrm{Frob}_2)) &= 2^a + 2^b (\bmod p) \\ &= 258573 (\bmod p). \end{aligned}$$

Using Stein's tables we see that $\mathrm{Tr}(\overline{\rho}_i(\mathrm{Frob}_2)) = \alpha_i$, so we must have that $\overline{\alpha}_i \equiv 258573 (\bmod \varpi)$. This also shows that $\overline{\alpha}_i$ must belong to $\mathbb{F}_p$ and so must be a root of one of the linear factors of $g(x)$ modulo p. Using Maple to compute the linear roots of $g(x)$ modulo p we find that they are 287487 and 85284, neither of which is congruent to 258573 modulo p. This provides a contradiction so we may conclude that $\overline{\rho}_i$ is irreducible.

Due to the size of the prime under consideration, it was not possible with the computer we used to compute the p^{th} Fourier coefficients of the f_i to check ordinarity. So, instead we show that in this case the ordinarity assumption is not necessary. We do this by showing there are no congruences between the f_i. Let E be a large number field containing all of the K_i. Let $i, j \in \{1, 2, 3, 4\}$ with $i \neq j$. Let $\mathfrak{q}$ be any prime of E over p. As in Section 5, f_i and f_j each give a map from $\mathbb{T}_{\mathcal{O}_{E_\mathfrak{q}}}$ to $\mathcal{O}_{E_\mathfrak{q}}$ given by $T(\ell) \mapsto a_{f_i}(\ell)$ and $T(\ell) \mapsto a_{f_j}(\ell)$ respectively. Let $\mathfrak{m}_i$ and $\mathfrak{m}_j$ be the respective maximal ideals defined as the inverse image of $\mathfrak{q}$ under these maps. (These are the maximal ideals associated to f_i and f_j of $\mathbb{T}_{\mathcal{O}_{E_\mathfrak{q}}}$ as in Section 5.) There is a congruence between f_i and f_j modulo $\mathfrak{q}$ if and only if the maximal ideals $\mathfrak{m}_i$ and $\mathfrak{m}_j$ are the same. This is equivalent to the statement that

$$a_{f_i}(\ell) \equiv a_{f_j}(\ell) (\bmod \mathfrak{q})$$

for all $\ell \neq p$. In particular, looking at the case when $\ell = 2$, if a congruence exists between f_i and f_j we have

$$\mathfrak{q} \mid (a_{f_i}(2) - a_{f_j}(2)),$$

i.e.,

$$\mathfrak{q} \mid (\alpha_i - \alpha_j).$$

Therefore we have that

$$\mathrm{Nm}(\mathfrak{q}) \mid \mathrm{Nm}(\alpha_i - \alpha_j).$$

The left hand side is a power of p where as the right hand side divides a power of the discriminant of $g(x)$, so that necessarily p divides the discriminant of $g(x)$.

Computing the discriminant with Maple we find the prime factorization of the discriminant,

$$\begin{aligned}\operatorname{disc}(g(x)) = &-2^{48}3^3 5^6 \cdot 11 \cdot 59 \cdot 15909926723 \cdot 4581597403 \\ &\cdot 61912455248726091228769884731066259290896074682396020673553.\end{aligned}$$

Therefore we have that p does not divide this discriminant. Therefore we must have that there is no congruence modulo $\mathfrak{q}$ between any of the f_j's. We can now appeal to the same argument used in the proof of Lemma 6.4 to conclude that there exists a Hecke operator t so that $t \cdot f_i = u \cdot \frac{\langle f_i, f_i \rangle}{\Omega_{f_i}^+ \Omega_{f_i}^-} f_i$ and $t \cdot f_j = 0$ for $j \neq i$. In this way we have avoided needing to check the ordinarity of each f_j to get the existence of the Hecke operator conjectured in Conjecture 5.2.

If we choose f_i to be one of the two newforms with real Fourier coefficients, then we satisfy all of the hypotheses of Theorem 8.4 and so obtain the result that

$$516223 \mid \# \mathrm{H}^1_f(\mathbb{Q}, W_{f_i, \wp_i}(-27)).$$

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